



# Bergman kernels and symplectic reduction

Xiaonan Ma, Weiping Zhang

## ► To cite this version:

| Xiaonan Ma, Weiping Zhang. Bergman kernels and symplectic reduction. 2006. hal-00087484

**HAL Id: hal-00087484**

**<https://hal.science/hal-00087484>**

Preprint submitted on 24 Jul 2006

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# BERGMAN KERNELS AND SYMPLECTIC REDUCTION

XIAONAN MA AND WEIPING ZHANG

**ABSTRACT.** We generalize several recent results concerning the asymptotic expansions of Bergman kernels to the framework of geometric quantization and establish an asymptotic symplectic identification property. More precisely, we study the asymptotic expansion of the  $G$ -invariant Bergman kernel of the  $\text{spin}^c$  Dirac operator associated with high tensor powers of a positive line bundle on a symplectic manifold. We also develop a way to compute the coefficients of the expansion, and compute the first few of them, especially, we obtain the scalar curvature of the reduction space from the  $G$ -invariant Bergman kernel on the total space. These results generalize the corresponding results in the non-equivariant setting, which has played a crucial role in the recent work of Donaldson on stability of projective manifolds, to the geometric quantization setting. As another kind of application, we generalize some Toeplitz operator type properties in semi-classical analysis to the framework of geometric quantization. The method we use is inspired by Local Index Theory, especially by the analytic localization techniques developed by Bismut and Lebeau.

## CONTENTS

0. Introduction	2
1. Connections and Laplacians associated to a principal bundle	13
1.1. Connections associated to a principal bundle	13
1.2. Curvatures and Laplacians associated to a principal bundle	15
2. $G$ -invariant Bergman kernels	17
2.1. Casimir operator	17
2.2. $\text{Spin}^c$ Dirac operator	18
2.3. $G$ -invariant Bergman kernel	20
2.4. Localization of the problem	22
2.5. Induced operator on $U/G$	26
2.6. Rescaling and a Taylor expansion of the operator $\Phi \mathcal{L}_p \Phi^{-1}$	30
2.7. Uniform estimate on the $G$ -invariant Bergman kernel	36
2.8. Evaluation of $J_{r,u}$	47
2.9. Proof of Theorem 0.2	48
3. Evaluation of $P^{(r)}$	50
3.1. Spectrum of $\mathcal{L}_2^0$	50
3.2. Evaluation of $P^{(r)}$ : a proof of (0.12) and (0.13)	53
3.3. A formula for $\mathcal{O}_1$	55
3.4. Example $(\mathbb{C}P^1, 2\omega_{FS})$	59
4. Applications	61
4.1. Orbifold case	62

4.2.	$\vartheta$ -weight Bergman kernel on $X$	64
4.3.	Averaging the Bergman kernel: a direct proof of (0.15) and (0.16)	65
4.4.	Toeplitz operators on $X_G$	68
4.5.	Generalization to non-compact manifolds	80
4.6.	Relation on the Bergman kernel on $X_G$	81
5.	Computing the coefficient $\Phi_1$ and $P^{(2)}(0, 0)$	83
5.1.	The second fundamental form of $P$	83
5.2.	Operators $\mathcal{O}_1, \mathcal{O}_2$ in (2.103)	86
5.3.	Computation of the coefficient $\Phi_1$	98
5.4.	Final computations: the proof of Theorem 0.6	108
5.5.	Coefficient $P^{(2)}(0, 0)$	110
5.6.	Coefficient $\Phi_1$ : general case	128
6.	Bergman kernel and geometric quantization	129
	Acknowledgments	131
	References	131

## 0. INTRODUCTION

The study of the Bergman kernel is a classical subject in the theory of several complex variables, where usually it concerns the kernel function of the projection operator to an infinite dimensional Hilbert space. The recent interest of the analogue of this concept in complex geometry mainly started in the paper of Tian [39], which was in turn inspired by a question of Yau. Here, the projection is onto a finite dimensional space instead.

After [39], the Bergman kernel has been studied extensively in [35], [42], [14], [23], establishing the diagonal asymptotic expansion for high powers of an ample line bundle. Moreover, the coefficients in the asymptotic expansion encode geometric information of the underlying complex projective manifolds. This asymptotic expansion plays a crucial role in the recent work of Donaldson [18], where the existence of Kähler metrics with constant scalar curvature is shown to be closely related to Chow-Mumford stability.

In [17], [26], [27], Dai, Liu, Ma and Marinescu studied the *full off-diagonal* asymptotic expansion of the (generalized) Bergman kernel of the  $\text{spin}^c$  Dirac operator and the renormalized Bochner–Laplacian associated to a positive line bundle on a compact symplectic manifold. As a by product, they gave a new proof of the results mentioned in the previous paragraph. They find also various applications therein, especially as pointed out in [27], the *full off-diagonal* asymptotic expansion implies Toeplitz operator type properties. This approach is inspired by the Local Index Theory, especially by the analytic localization techniques of Bismut-Lebeau [7, §11]. We refer to the above papers and the recent book [28] for detail informations of the Bergman kernel on compact symplectic manifolds.

In this paper, we generalize some of the results in [17], [26] and [27] to the framework of geometric quantization, by studying the asymptotic expansion of the  $G$ -invariant

Bergman kernel for high powers of an ample line bundle on symplectic manifolds admitting a Hamiltonian group action.

To start with, let  $(X, \omega)$  be a compact symplectic manifold of real dimension  $2n$ . Assume that there exists a Hermitian line bundle  $L$  over  $X$  endowed with a Hermitian connection  $\nabla^L$  with the property that

$$(0.1) \quad \frac{\sqrt{-1}}{2\pi} R^L = \omega,$$

where  $R^L = (\nabla^L)^2$  is the curvature of  $(L, \nabla^L)$ .

Let  $(E, h^E)$  be a Hermitian vector bundle on  $X$  equipped with a Hermitian connection  $\nabla^E$  and  $R^E$  denotes the associated curvature.

Let  $g^{TX}$  be a Riemannian metric on  $X$ . Let  $\mathbf{J} : TX \rightarrow TX$  be the skew-adjoint linear map which satisfies the relation

$$(0.2) \quad \omega(u, v) = g^{TX}(\mathbf{J}u, v)$$

for  $u, v \in TX$ .

Let  $J$  be an almost complex structure such that

$$(0.3) \quad g^{TX}(Ju, Jv) = g^{TX}(u, v), \quad \omega(Ju, Jv) = \omega(u, v)$$

and that  $\omega(\cdot, J\cdot)$  defines a metric on  $TX$ . Then  $J$  commutes with  $\mathbf{J}$  and  $J = \mathbf{J}(-\mathbf{J}^2)^{-1/2}$  (cf. (2.7)).

Let  $\nabla^{TX}$  be the Levi-Civita connection on  $(TX, g^{TX})$  with curvature  $R^{TX}$  and scalar curvature  $r^X$ . The connection  $\nabla^{TX}$  induces a natural connection  $\nabla^{\det}$  on  $\det(T^{(1,0)}X)$  with curvature  $R^{\det}$ , and the Clifford connection  $\nabla^{\text{Cliff}}$  on the Clifford module  $\Lambda(T^{*(0,1)}X)$  with curvature  $R^{\text{Cliff}}$  (cf. Section 2.2).

The  $\text{spin}^c$  Dirac operator  $D_p$  acts on  $\Omega^{0,\bullet}(X, L^p \otimes E) = \bigoplus_{q=0}^n \Omega^{0,q}(X, L^p \otimes E)$ , the direct sum of spaces of  $(0, q)$ -forms with values in  $L^p \otimes E$ . We denote by  $D_p^+$  the restriction of  $D_p$  on  $\Omega^{0,\text{even}}(X, L^p \otimes E)$ . The index of  $D_p^+$  is defined by

$$(0.4) \quad \text{Ind}(D_p^+) = \text{Ker } D_p^+ - \text{Coker } D_p^+.$$

Let  $G$  be a compact connected Lie group with Lie algebra  $\mathfrak{g}$  and  $\dim G = n_0$ . Suppose that  $G$  acts on  $X$  and its action on  $X$  lifts on  $L$  and  $E$ . Moreover, we assume the  $G$ -action preserves the above connections and metrics on  $TX, L, E$  and  $J$ . Then  $\text{Ind}(D_p^+)$  is a virtual representation of  $G$ . Denote by  $(\text{Ker } D_p)^G, \text{Ind}(D_p^+)^G$  the  $G$ -trivial components of  $\text{Ker } D_p, \text{Ind}(D_p^+)$  respectively.

The action of  $G$  on  $L$  induces naturally a moment map  $\mu : X \rightarrow \mathfrak{g}^*$  (cf. (2.14)). We assume that  $0 \in \mathfrak{g}^*$  is a regular value of  $\mu$ .

Set  $P = \mu^{-1}(0)$ . Then the Marsden-Weinstein symplectic reduction  $(X_G = P/G, \omega_{X_G})$  is a symplectic orbifold ( $X_G$  is smooth if  $G$  acts freely on  $P$ ).

Moreover,  $(L, \nabla^L), (E, \nabla^E)$  descend to  $(L_G, \nabla^{L_G}), (E_G, \nabla^{E_G})$  over  $X_G$  so that the corresponding curvature condition  $\frac{\sqrt{-1}}{2\pi} R^{L_G} = \omega_G$  holds (cf. [20]). The  $G$ -invariant almost complex structure  $J$  also descends to an almost complex structure  $J_G$  on  $TX_G$ , and  $h^L, h^E, g^{TX}$  descend to  $h^{L_G}, h^{E_G}, g^{TX_G}$  respectively.

One can construct the corresponding  $\text{spin}^c$  Dirac operator  $D_{G,p}$  on  $X_G$ .

The geometric quantization conjecture of Guillemin-Sternberg [20] can be stated as follows: when  $E$  is the trivial bundle  $\mathbb{C}$  on  $X$ , for any  $p > 0$ ,

$$(0.5) \quad \dim(\operatorname{Ind}(D_p^+)^G) = \dim(\operatorname{Ind}(D_{G,p}^+)).$$

When  $G$  is abelian, this conjecture was proved by Meinrenken [31] and Vergne [41]. The remaining nonabelian case was proved by Meinrenken [32] using the symplectic cut techniques of Lerman, and by Tian and Zhang [40] using analytic localization techniques.

More generally, by a result of Tian and Zhang [40, Theorem 0.2], for any general vector bundle  $E$  as above, there exists  $p_0 > 0$  such that for any  $p \geq p_0$ , (0.5) still holds.

On the other hand, by [25, Theorem 2.5] (cf. (2.13)), which is a direct consequence of the Lichnerowicz formula for  $D_p$ , for  $p$  large enough,  $\operatorname{Coker} D_p^+$  is null (cf. also [10], [13]). Thus there exists  $p_0 > 0$  such that for any  $p \geq p_0$ ,

$$(0.6) \quad \begin{aligned} \dim(\operatorname{Ker} D_p)^G &= \dim(\operatorname{Ker} D_{G,p}) = \dim(\operatorname{Ind}(D_{G,p}^+)) = \int_{X_G} \operatorname{Td}(TX_G) \operatorname{ch}(L_G^p \otimes E_G) \\ &= \operatorname{rk}(E) \int_{X_G} \frac{(p c_1(L_G))^{n-n_0}}{(n-n_0)!} + \int_{X_G} \left( c_1(E_G) + \frac{\operatorname{rk}(E)}{2} c_1(TX_G) \right) \frac{(p c_1(L_G))^{n-n_0-1}}{(n-n_0-1)!} \\ &\quad + \mathcal{O}(p^{n-n_0-2}), \end{aligned}$$

where  $\operatorname{ch}(\cdot)$ ,  $c_1(\cdot)$ ,  $\operatorname{Td}(\cdot)$  are the Chern character, the first Chern class and the Todd class of the corresponding complex vector bundles ( $TX_G$  is a complex vector bundle with complex structure  $J_G$ ).

Set  $E_p := \Lambda(T^{*(0,1)}X) \otimes L^p \otimes E$ . Let  $\langle \cdot, \cdot \rangle$  be the  $L^2$ -scalar product on  $\Omega^{0,\bullet}(X, L^p \otimes E) = \mathcal{C}^\infty(X, E_p)$  induced by  $g^{TX}, h^L, h^E$  as in (1.19).

Let  $P_p^G$  be the orthogonal projection from  $(\Omega^{0,\bullet}(X, L^p \otimes E), \langle \cdot, \cdot \rangle)$  on  $(\operatorname{Ker} D_p)^G$ . The  $G$ -invariant Bergman kernel is  $P_p^G(x, x')$  ( $x, x' \in X$ ), the smooth kernel of  $P_p^G$  with respect to the Riemannian volume form  $dv_X(x')$ .

Let  $\operatorname{pr}_1$  and  $\operatorname{pr}_2$  be the projections from  $X \times X$  onto the first and second factor  $X$  respectively. Then  $P_p^G(x, x')$  is a smooth section of  $\operatorname{pr}_1^*(E_p) \otimes \operatorname{pr}_2^*(E_p^*)$  on  $X \times X$ . In particular,  $P_p^G(x, x) \in \operatorname{End}(E_p)_x = \operatorname{End}(\Lambda(T^{*(0,1)}X) \otimes E)_x$ .

The  $G$ -invariant Bergman kernel  $P_p^G(x, x')$  is an analytic version of  $(\operatorname{Ker} D_p)^G$ . In view of (0.6), it is natural to expect that the kernel  $P_p^G(x, x')$  should be closely related to the corresponding Bergman kernel on the symplectic reduction  $X_G$ . The purpose of this paper is to study the asymptotic expansion of the  $G$ -invariant Bergman kernel  $P_p^G(x, x')$  as  $p \rightarrow \infty$ , and we will relate it to the asymptotic expansion of the Bergman kernel on the symplectic reduction  $X_G$ .

Let  $d^X(x, x')$  be the Riemannian distance between  $x, x' \in X$ .

In Section 2.4, we prove the following result which allows us to reduce our problem as a problem near  $P = \mu^{-1}(0)$ .

**Theorem 0.1.** *For any open  $G$ -neighborhood  $U$  of  $P$  in  $X$ ,  $\varepsilon_0 > 0$ ,  $l, m \in \mathbb{N}$ , there exists  $C_{l,m} > 0$  (depend on  $U, \varepsilon_0$ ) such that for  $p \geq 1$ ,  $x, x' \in X, d^X(Gx, x') \geq \varepsilon_0$  or*

$x, x' \in X \setminus U$ ,

$$(0.7) \quad |P_p^G(x, x')|_{\mathcal{C}^m} \leq C_{l,m} p^{-l},$$

where  $\mathcal{C}^m$  is the  $\mathcal{C}^m$ -norm induced by  $\nabla^L, \nabla^E, \nabla^{TX}, h^L, h^E$  and  $g^{TX}$ .

Assume for simplicity that  $G$  acts freely on  $P$ .

Let  $U$  be an open  $G$ -neighborhood of  $\mu^{-1}(0)$  such that  $G$  acts freely on  $U$ .

For any  $G$ -equivariant vector bundle  $(F, \nabla^F)$  on  $U$ , we denote by  $F_B$  the bundle on  $U/G = B$  induced naturally by  $G$ -invariant sections of  $F$  on  $U$ . The connection  $\nabla^F$  induces canonically a connection  $\nabla^{F_B}$  on  $F_B$ . Let  $R^{F_B}$  be its curvature. Let

$$(0.8) \quad \mu^F(K) = \nabla_{K^X}^F - L_K \in \text{End}(F)$$

for  $K \in \mathfrak{g}$  and  $K^X$  the corresponding vector field on  $U$ .

Note that  $P_p^G \in (\mathcal{C}^\infty(U \times U, \text{pr}_1^* E_p \otimes \text{pr}_2^* E_p^*))^{G \times G}$ , thus we can view  $P_p^G(x, x')$  ( $x, x' \in U$ ) as a smooth section of  $\text{pr}_1^*(E_p)_B \otimes \text{pr}_2^*(E_p^*)_B$  on  $B \times B$ .

Let  $g^{TB}$  be the Riemannian metric on  $U/G = B$  induced by  $g^{TX}$ . Let  $\nabla^{TB}$  be the Levi-Civita connection on  $(TB, g^{TB})$  with curvature  $R^{TB}$ . Let  $N_G$  be the normal bundle to  $X_G$  in  $B$ . We identify  $N_G$  with the orthogonal complement of  $TX_G$  in  $(TB|_{X_G}, g^{TB})$ .

Let  $g^{TX_G}, g^{N_G}$  be the metrics on  $TX_G, N_G$  induced by  $g^{TB}$  respectively.

Let  $P^{TX_G}, P^{N_G}$  be the orthogonal projections from  $TB|_{X_G}$  on  $TX_G, N_G$  respectively. Set

$$(0.9) \quad \begin{aligned} \nabla^{N_G} &= P^{N_G}(\nabla^{TB}|_{X_G})P^{N_G}, \quad \nabla^{TX_G} = P^{TX_G}(\nabla^{TB}|_{X_G})P^{TX_G}, \\ {}^0\nabla^{TB} &= \nabla^{TX_G} \oplus \nabla^{N_G}, \quad A = \nabla^{TB}|_{X_G} - {}^0\nabla^{TB}. \end{aligned}$$

Then  $\nabla^{N_G}, {}^0\nabla^{TB}$  are Euclidean connections on  $N_G, TB|_{X_G}$  on  $X_G$ ,  $\nabla^{TX_G}$  is the Levi-Civita connection on  $(TX_G, g^{TX_G})$ , and  $A$  is the associated second fundamental form.

Denote by  $\text{vol}(Gx)$  ( $x \in U$ ) the volume of the orbit  $Gx$  equipped with the metric induced by  $g^{TX}$ . Following [40, (3.10)], let  $h(x)$  be the function on  $U$  defined by

$$(0.10) \quad h(x) = (\text{vol}(Gx))^{1/2}.$$

Then  $h$  reduces to a function on  $B$ .

Denote by  $I_{\mathbb{C} \otimes E}$  the projection from  $\Lambda(T^{*(0,1)}X) \otimes E$  onto  $\mathbb{C} \otimes E$  under the decomposition  $\Lambda(T^{*(0,1)}X) \otimes E = \mathbb{C} \otimes E \oplus \Lambda^{>0}(T^{*(0,1)}X) \otimes E$ , and  $I_{\mathbb{C} \otimes E_B}$  the corresponding projection on  $B$ .

In the whole paper, for any  $x_0 \in X_G, Z \in T_{x_0}B$ , we write  $Z = Z^0 + Z^\perp$ , with  $Z^0 \in T_{x_0}X_G, Z^\perp \in N_{G, x_0}$ .

Let  $\tau_{Z^0} Z^\perp \in N_{G, \exp_{x_0}^{X_G}(Z^0)}$  be the parallel transport of  $Z^\perp$  with respect to the connection  $\nabla^{N_G}$  along the geodesic in  $X_G, [0, 1] \ni t \rightarrow \exp_{x_0}^{X_G}(tZ^0)$ .

For  $\varepsilon_0 > 0$  small enough, we identify  $Z \in T_{x_0}B, |Z| < \varepsilon_0$  with  $\exp_{\exp_{x_0}^{X_G}(Z^0)}^B(\tau_{Z^0} Z^\perp) \in B$ . Then for  $x_0 \in X_G, Z, Z' \in T_{x_0}B, |Z|, |Z'| < \varepsilon_0$ , the map  $\Psi : TB|_{X_G} \times TB|_{X_G} \rightarrow B \times B$ ,

$$\Psi(Z, Z') = (\exp_{\exp_{x_0}^{X_G}(Z^0)}^B(\tau_{Z^0} Z^\perp), \exp_{\exp_{x_0}^{X_G}(Z'^0)}^B(\tau_{Z'^0} Z'^\perp))$$

is well defined.

We identify  $(E_p)_{B,Z}$  to  $(E_p)_{B,x_0}$  by using parallel transport with respect to  $\nabla^{(E_p)_B}$  along  $[0, 1] \ni u \rightarrow uZ$ .

Let  $\pi_B : TB|_{X_G} \times TB|_{X_G} \rightarrow X_G$  be the natural projection from the fiberwise product of  $TB|_{X_G}$  on  $X_G$  onto  $X_G$ .

From Theorem 0.1, we only need to understand  $P_p^G \circ \Psi$ , and under our identification,  $P_p^G \circ \Psi(Z, Z')$  is a smooth section of

$$\pi_B^*(\text{End}(E_p)_B) = \pi_B^*(\text{End}(\Lambda(T^{*(0,1)}X) \otimes E)_B)$$

on  $TB|_{X_G} \times TB|_{X_G}$ .

Let  $|\cdot|_{\mathcal{C}^{m'}(X_G)}$  be the  $\mathcal{C}^{m'}$ -norm on  $\mathcal{C}^\infty(X_G, \text{End}(\Lambda(T^{*(0,1)}X) \otimes E)_B)$  induced by  $\nabla^{\text{Cliff}_B}, \nabla^{E_B}, h^E$  and  $g^{TX}$ . The norm  $|\cdot|_{\mathcal{C}^{m'}(X_G)}$  induces naturally a  $\mathcal{C}^{m'}$ -norm along  $X_G$  on  $\mathcal{C}^\infty(TB|_{X_G} \times TB|_{X_G}, \pi_B^*(\text{End}(\Lambda(T^{*(0,1)}X) \otimes E)_B))$ , we still denote it by  $|\cdot|_{\mathcal{C}^{m'}(X_G)}$ .

Let  $dv_B, dv_{X_G}, dv_{N_G}$  be the Riemannian volume forms on  $(B, g^{TB}), (X_G, g^{TX_G}), (N_G, g^{N_G})$  respectively. Let  $\kappa \in \mathcal{C}^\infty(TB|_{X_G}, \mathbb{R})$ , with  $\kappa = 1$  on  $X_G$ , be defined by that for  $Z \in T_{x_0}B$ ,  $x_0 \in X_G$ ,

$$(0.11) \quad dv_B(x_0, Z) = \kappa(x_0, Z)dv_{T_{x_0}B}(Z) = \kappa(x_0, Z)dv_{X_G}(x_0)dv_{N_G, x_0}.$$

The following result is one of the main results of this paper.

**Theorem 0.2.** *Assume that  $G$  acts freely on  $\mu^{-1}(0)$  and  $\mathbf{J} = J$  on  $\mu^{-1}(0)$ . Then there exist  $\mathcal{Q}_r(Z, Z') \in \text{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{B, x_0}$  ( $x_0 \in X_G, r \in \mathbb{N}$ ), polynomials in  $Z, Z'$  with the same parity as  $r$ , whose coefficients are polynomials in  $A, R^{TB}, R^{\text{Cliff}_B}, R^{E_B}, \mu^E, \mu^{\text{Cliff}}$  (resp.  $r^X, R^{\det}, R^E$ ; resp.  $h, R^L, R^{L_B}$ ; resp.  $\mu$ ) and their derivatives at  $x_0$  to order  $r-1$  (resp.  $r-2$ ; resp.  $r$ , resp.  $r+1$ ), such that if we denote by*

$$(0.12) \quad P_{x_0}^{(r)}(Z, Z') = \mathcal{Q}_r(Z, Z')P(Z, Z'), \quad \mathcal{Q}_0(Z, Z') = I_{\mathbb{C} \otimes E_B},$$

with

$$(0.13) \quad P(Z, Z') = \exp\left(-\frac{\pi}{2}|Z^0 - Z'^0|^2 - \pi\sqrt{-1}\langle J_{x_0}Z^0, Z'^0 \rangle\right) \\ \times 2^{\frac{n_0}{2}} \exp\left(-\pi(|Z^\perp|^2 + |Z'^\perp|^2)\right),$$

then there exists  $C'' > 0$  such that for any  $k, m, m', m'' \in \mathbb{N}$ , there exists  $C > 0$  such that for  $x_0 \in X_G, Z, Z' \in T_{x_0}B, |Z|, |Z'| \leq \varepsilon_0$ ,<sup>1</sup>

$$(0.14) \quad (1 + \sqrt{p}|Z^\perp| + \sqrt{p}|Z'^\perp|)^{m''} \sup_{|\alpha|+|\alpha'| \leq m} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \right| \\ \left( p^{-n+\frac{n_0}{2}} (h\kappa^{\frac{1}{2}})(Z)(h\kappa^{\frac{1}{2}})(Z')P_p^G \circ \Psi(Z, Z') - \sum_{r=0}^k P_{x_0}^{(r)}(\sqrt{p}Z, \sqrt{p}Z')p^{-\frac{r}{2}} \right) \Big|_{\mathcal{C}^{m'}(X_G)} \\ \leq Cp^{-(k+1-m)/2} (1 + \sqrt{p}|Z^0| + \sqrt{p}|Z'^0|)^{2(n+k+m'+2)+m} \exp(-\sqrt{C''}\sqrt{p}|Z - Z'|) + \mathcal{O}(p^{-\infty}).$$

Furthermore, the expansion is uniform in the following sense: for any fixed  $k, m, m', m'' \in \mathbb{N}$ , assume that the derivatives of  $g^{TX}, h^L, \nabla^L, h^E, \nabla^E$ , and  $J$  with order  $\leq 2n + k +$

<sup>1</sup>In the exponential factor of [29, (7)], we missed  $m'$  as in the last line of (0.14) here.

$m + m' + 3$  run over a set bounded in the  $\mathcal{C}^{m'}$ -norm taken with respect to the parameters and, moreover,  $g^{TX}$  runs over a set bounded below. Then the constant  $C$  is independent of  $g^{TX}$ ; and the  $\mathcal{C}^{m'}$ -norm in (0.14) includes also the derivatives on the parameters.

In (0.14), the term  $\mathcal{O}(p^{-\infty})$  means that for any  $l, l_1 \in \mathbb{N}$ , there exists  $C_{l,l_1} > 0$  such that its  $\mathcal{C}^{l_1}$ -norm is dominated by  $C_{l,l_1} p^{-l}$ .

It is interesting to see that the kernel  $P(Z, Z')$  is the product of two kernels : along  $T_{x_0}X_G$ , it is the classical Bergman kernel on  $T_{x_0}X_G$  with complex structure  $J_{x_0}$ , while along  $N_G$ , it is the kernel of a harmonic oscillator on  $N_{G,x_0}$ .

**Remark 0.3.** i) Theorem 0.2 is a special case of Theorem 2.23 where we do not assume  $\mathbf{J} = J$  on  $P = \mu^{-1}(0)$ . In Theorem 3.2, we get explicit informations on  $P^{(r)}$  when  $\mathbf{J}$  verifies (3.2).

ii) If  $G$  does not act freely on  $P$ , then  $X_G$  is an orbifold. In Section 4.1, we explain how to modify our arguments to get the asymptotic expansion, Theorem 4.1. Analogous to the usual orbifold case [17, (5.27)],  $P_p^G(x, x)(x \in P)$  does not have a uniform asymptotic expansion if the singular set of  $X_G$  is not empty.

iii) Let  $\mathcal{V}$  be an irreducible representation of  $G$ , let  $P_p^\mathcal{V}$  be the orthogonal projection from  $\Omega^{0,\bullet}(X, L^p \otimes E)$  on  $\text{Hom}_G(\mathcal{V}, \text{Ker } D_p) \otimes \mathcal{V} \subset \text{Ker } D_p$ . In Section 4.2, we get the asymptotic expansion of the kernel  $P_p^\mathcal{V}(x, x')$  from Theorems 0.1, 0.2.

iv) When  $G = \{1\}$ , Theorem 0.2 is [17, Theorem 4.18'].

v) If we take  $Z = Z' = 0$  in (0.14), then we get for  $x_0 \in X_G$ ,

$$(0.15) \quad P_{x_0}^{(0)}(0, 0) = 2^{\frac{n_0}{2}} I_{\mathbb{C} \otimes E_B},$$

and

$$(0.16) \quad \left| p^{-n+\frac{n_0}{2}} h^2(x_0) P_p^G(x_0, x_0) - \sum_{r=0}^k P_{x_0}^{(2r)}(0, 0) p^{-r} \right|_{\mathcal{C}^{m'}(X_G)} \leq C p^{-k-1}.$$

In Section 4.3, we show that (0.15) and (0.16) are direct consequences of the full off-diagonal asymptotic expansion of the Bergman kernel [17, Theorem 4.18']. In fact, one possible way to get Theorem 0.2 is to average the full off-diagonal asymptotic expansion of the Bergman kernel on  $X$  [17, Theorem 4.18'] with respect to a Haar measure on  $G$ . However, we do not know how to get the full off-diagonal expansion, especially the fast decay along  $N_G$  in (0.14) in this way.

In this paper we will apply the analytic localization techniques to get Theorem 0.2, and this method also gives us an effective way to compute the coefficients in the asymptotic expansion (cf. §3.2). The key observation is that the  $G$ -invariant Bergman kernel is exactly the kernel of the orthogonal projection to the zero space of a deformation of  $D_p^2$  by the Casimir operator (i.e., to consider  $D_p^2 - p\text{Cas}$ ). This plays an essential role in proving Theorems 0.1, 0.2.

Let  $\mathcal{S}_p$  be a section of  $\text{End}(\Lambda(T^{*(0,1)}X) \otimes E)_B$  on  $X_G$  defined by

$$(0.17) \quad \mathcal{S}_p(x_0) = \int_{Z \in N_G, |Z| \leq \varepsilon_0} h^2(x_0, Z) P_p^G \circ \Psi((x_0, Z), (x_0, Z)) \kappa(x_0, Z) dv_{N_G}(Z).$$



By Theorem 0.1, modulo  $\mathcal{O}(p^{-\infty})$ ,  $\mathcal{J}_p(x_0)$  does not depend on  $\varepsilon_0$ , and

$$\begin{aligned}
 (0.18) \quad \dim(\text{Ker } D_p)^G &= \int_X \text{Tr}[P_p^G(y, y)] dv_X(y) = \int_U \text{Tr}[P_p^G(y, y)] dv_X(y) + \mathcal{O}(p^{-\infty}) \\
 &= \int_B h^2(y) \text{Tr}[P_p^G(y, y)] dv_B(y) + \mathcal{O}(p^{-\infty}) \\
 &= \int_{X_G} \text{Tr}[\mathcal{J}_p(x_0)] dv_{X_G}(x_0) + \mathcal{O}(p^{-\infty}).
 \end{aligned}$$

A direct consequence of Theorem 0.2 is the following corollary.

**Corollary 0.4.** *Taken  $Z = Z' \in N_{G, x_0}$ ,  $m = 0$  in (0.14), we get*

$$\begin{aligned}
 (0.19) \quad \left| p^{-n+\frac{n_0}{2}} (h^2 \kappa)(Z) P_p^G(Z, Z) - \sum_{r=0}^k P_{x_0}^{(r)}(\sqrt{p}Z, \sqrt{p}Z) p^{-r/2} \right|_{\mathcal{C}^{m'}(X_G)} \\
 \leq C p^{-(k+1)/2} (1 + \sqrt{p}|Z|)^{-m'} + \mathcal{O}(p^{-\infty}).
 \end{aligned}$$

In particular, there exist  $\Phi_r \in \text{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{B, x_0}$  ( $r \in \mathbb{N}$ ) which are polynomials in  $A$ ,  $R^{TB}$ ,  $R^{\text{Cliff}_B}$ ,  $R^{E_B}$ ,  $\mu^E$ ,  $\mu^{\text{Cliff}}$ , (resp.  $r^X$ ,  $R^{\det}$ ,  $R^E$ ; resp.  $h$ ,  $R^{L_B}$ ,  $R^L$ ; resp.  $\mu$ ), and their derivatives at  $x_0$  up to order  $2r-1$  (resp.  $2r-2$ ; resp.  $2r$ ; resp.  $2r+1$ ), and  $\Phi_0 = I_{\mathbb{C} \otimes E_B}$ , such that for any  $k, m' \in \mathbb{N}$ , there exists  $C_{k, m'} > 0$  such that for any  $x_0 \in X_G$ ,  $p \in \mathbb{N}$ ,

$$(0.20) \quad \left| p^{-n+n_0} \mathcal{J}_p(x_0) - \sum_{r=0}^k \Phi_r(x_0) p^{-r} \right|_{\mathcal{C}^{m'}} \leq C_{k, m'} p^{-k-1}.$$

In the rest of Introduction, we will specify our results in the Kähler case.

We suppose now that  $(X, \omega, J)$  is a compact Kähler manifold and  $\mathbf{J} = J$  on  $X$ . Assume also that  $(L, h^L, \nabla^L)$ ,  $(E, h^E, \nabla^E)$  are holomorphic Hermitian vector bundles with holomorphic Hermitian connections, and the action of  $G$  on  $X, L, E$  is holomorphic.

Let  $\bar{\partial}^{L^p \otimes E, *}$  be the formal adjoint of the Dolbeault operator  $\bar{\partial}^{L^p \otimes E}$ , then

$$(0.21) \quad D_p = \sqrt{2}(\bar{\partial}^{L^p \otimes E} + \bar{\partial}^{L^p \otimes E, *}),$$

and

$$(0.22) \quad D_p^2 = 2 \left( \bar{\partial}^{L^p \otimes E} \bar{\partial}^{L^p \otimes E, *} + \bar{\partial}^{L^p \otimes E, *} \bar{\partial}^{L^p \otimes E} \right)$$

preserves the  $\mathbb{Z}$ -grading of  $\Omega^{0, \bullet}(X, L^p \otimes E)$ .

By the Kodaira vanishing theorem, for  $p$  large enough,

$$(0.23) \quad (\text{Ker } D_p)^G = H^0(X, L^p \otimes E)^G.$$

Thus for  $p$  large enough,  $P_p^G(x, x') \in (L^p \otimes E)_x \otimes (L^p \otimes E)_{x'}^*$ , and so  $P_p^G(x, x) \in \text{End}(E_x)$ ,  $\mathcal{J}_p(x_0) \in \text{End}(E_{x_0})$ . In particular, in (0.15),

$$(0.24) \quad P_{x_0}^{(0)}(0, 0) = 2^{\frac{n_0}{2}} \text{Id}_{E_G}.$$

**Remark 0.5.** In the special case of  $E = \mathbb{C}$ ,  $P_p^G(x_0, x_0)$  is a non-negative function on  $X_G$ , and (0.16) has been proved in [33, Theorem 1] without knowing the informations on  $P_{x_0}^{(2r)}(0, 0)$ , while in [34, Theorem 1], it was claimed that  $P_{x_0}^{(0)}(0, 0) = 1$ . In [33, Prop. 1], Paoletti knew that for any  $l \in \mathbb{N}$ , there is  $C > 0$  such that for any  $p$ ,  $|P_p^G(x, x)| \leq Cp^{-l}$  uniformly on any compact subset of  $X \setminus (\mu^{-1}(0) \cup R)$ , with  $R$  the subset of unstable points of the action of  $G$ . In [34], some Toeplitz operator type properties on  $X_G$  was also claimed from the analysis of Toeplitz structures of Boutet de Monvel–Guillemin [11], Boutet de Monvel–Sjöstrand [12] and Shiffman–Zelditch [36]. If we suppose moreover that  $G$  is a torus, Charles [15] has also a different version on the Toeplitz operator type properties on  $X_G$ .

In Section 4.4, we will show that Theorem 0.2 implies properties of Toeplitz operators on  $X_G$  (which also hold in the symplectic case). In particular, we recover the results on Toeplitz operators [15], [34].

Let  $\tilde{h}$  denote the restriction to  $X_G$  of the function  $h$  defined in (0.10).

The second main result of this paper is that we can in fact obtain the scalar curvature  $r^{X_G}$  on the symplectic reduction  $X_G$  from  $\mathcal{I}_p$ .

**Theorem 0.6.** *If  $(X, \omega)$  is a compact Kähler manifold and  $L, E$  are holomorphic vector bundles with holomorphic Hermitian connections  $\nabla^L, \nabla^E$ ,  $\mathbf{J} = J$ , and  $G$  acts freely on  $\mu^{-1}(0)$ , then for  $p$  large enough,  $\mathcal{I}_p(x_0) \in \text{End}(E_G)_{x_0}$ , and in (0.20),  $\Phi_r(x_0) \in \text{End}(E_G)_{x_0}$  are polynomials in  $A, R^{TB}, R^{EB}, \mu^E, R^E$  (resp.  $h, R^{LB}$ ; resp.  $\mu$ ) and their derivatives at  $x_0$  to order  $2r - 1$  (resp.  $2r$ , resp.  $2r + 1$ ), and  $\Phi_0 = \text{Id}_{E_G}$ . Moreover*

$$(0.25) \quad \Phi_1(x_0) = \frac{1}{8\pi} r^{X_G} + \frac{3}{4\pi} \Delta_{X_G} \log \tilde{h} + \frac{1}{2\pi} R_{x_0}^{E_G}(w_j^0, \bar{w}_j^0).$$

Here  $r^{X_G}$  is the Riemannian scalar curvature of  $(TX_G, g^{TX_G})$ ,  $\Delta_{X_G}$  is the Bochner-Laplacian on  $X_G$  (cf. (1.21)), and  $\{w_j^0\}$  is an orthonormal basis of  $T^{(1,0)}X_G$ .

Since the non-equivariant version of this result has already played a crucial role in the work of Donaldson mentioned before, we have reason to believe that Theorem 0.6 might also play a role in the study of stability of projective manifolds. Indeed, as Donaldson usually interprets his results in the framework of geometric quantization, this seems likely to be so.

We recover (0.6) from (0.25) after taking the trace, and then the integration on  $X_G$ . Thus (0.25) is a local version of (0.6) in the spirit of the Local Index Theory. The appearance of the term  $\frac{3}{4\pi} \Delta_{X_G} \log \tilde{h}$  is unexpected.

Let  $T$  be the torsion of the connection  ${}^0\nabla^{TX}$  in (1.2) on  $U$ . The curvature  $\Theta$  of the principal bundle  $U \rightarrow B$  relates to the torsion  $T$  by (1.6).

Following (3.6) and (5.20), we choose  $\{e_j^\perp\}$  to be an orthonormal basis of  $N_{G, x_0}$  and  $\{\frac{\partial}{\partial z_j^0}\} \in T_{x_0}^{(1,0)}X_G$  to be the holomorphic basis of the normal coordinate on  $X_G$ , and define  $\mathcal{T}_{klm}, \tilde{\mathcal{T}}_{jkl}$  as in (5.14). In particular, by Remark 5.3,  $\tilde{\mathcal{T}}_{jkl} = 0$  if  $G$  is abelian.

The  $G$ -invariant section  $\tilde{\mu}^E$  of  $TY \otimes \text{End}(E)$  on  $U$  is defined by (1.13) and (1.14).

If there is no other specific notification in the next formula (0.26), when we meet the operation  $|\cdot|^2$ , we will first do this operation, then take the sum of the indices.

**Theorem 0.7.** *Under the assumption of Theorem 0.6, for  $p > 0$  large enough,  $P_p^G(x, x) \in \text{End}(E_x)$  and  $P_{x_0}^{(r)}(0, 0) \in \text{End}(E_{x_0})$ . Moreover,*

$$(0.26) \quad P_{x_0}^{(2)}(0, 0) = 2^{\frac{n_0}{2}} \left\{ \frac{1}{8\pi} r_{x_0}^{X_G} + \frac{1}{\pi} R^{E_G} \left( \frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0} \right) + \frac{1}{\pi} \Delta_{X_G} \log \tilde{h} \right. \\ - \frac{3}{8\pi} \nabla_{e_k^\perp} \nabla_{e_k^\perp} \log h - \frac{2}{\pi} \sqrt{-1} \nabla_{JT \left( \frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_l^0} \right)} \log h - \frac{3}{\pi} \left| \nabla_{\frac{\partial}{\partial \bar{z}_j^0}} \log h \right|^2 \\ - \frac{5}{4\pi} \left| \nabla_{e_j^\perp} \log h \right|^2 + \frac{1}{2\pi} |T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})|^2 - \frac{1}{2\pi} \left| \sum_j T \left( \frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0} \right) \right|^2 \\ + \frac{1}{2\pi} \left| T \left( \frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0} \right) \right|^2 + \frac{1}{24\pi} \mathcal{T}_{klm}^2 + \frac{1}{26\pi} \tilde{\mathcal{T}}_{ijk} (3\tilde{\mathcal{T}}_{kji} - \tilde{\mathcal{T}}_{ijk}) + \frac{1}{2\pi} \langle \tilde{\mu}_{x_0}^E, \tilde{\mu}_{x_0}^E \rangle_{g^{TY}} \\ \left. + \frac{1}{\pi} \left\langle \tilde{\mu}^E, T \left( \frac{\partial}{\partial z_l^0}, \frac{\partial}{\partial \bar{z}_l^0} \right) \right\rangle + \frac{3\sqrt{-1}}{2\pi} \langle \tilde{\mu}^E, J e_k^\perp \rangle \nabla_{e_k^\perp} \log h + \frac{\sqrt{-1}}{4\pi} \left\langle J e_k^\perp, \nabla_{e_k^\perp}^{TY} \tilde{\mu}^E \right\rangle \right\}.$$

**Remark 0.8.** Certainly, if we only assume that  $\mathbf{J} = J$  on  $U$ , a neighborhood of  $P = \mu^{-1}(0)$ , then we still have  $\Phi_r(x_0) \in \text{End}(E_G)_{x_0}$ , as we work on the kernel of the Dirac operator  $D_p$ . Set  $\mathcal{S}_{p,0} = I_{\mathbb{C} \otimes E_G} \mathcal{S}_p I_{\mathbb{C} \otimes E_G}$ , the component of  $\mathcal{S}_p$  on  $\mathbb{C} \otimes E_G$ . As the computation is local, we still have Theorem 0.6 with  $\mathcal{S}_p$  replaced by  $\mathcal{S}_{p,0}$  and  $\mathcal{S}_p - \mathcal{S}_{p,0} = \mathcal{O}(p^{-\infty})$  (cf. (5.18)). If we only work on the  $\bar{\partial}$ -operator, i.e. the holomorphic sections, in Section 5.6, we explain how to reduce the case of general  $\mathbf{J}$  to the case  $\mathbf{J} = J$ . Same remark holds for  $P_p^G(x_0, x_0)$ .

Let  $i : P \hookrightarrow X$  be the natural injection.

Let  $\pi_G : \mathcal{C}^\infty(P, L^p \otimes E)^G \rightarrow \mathcal{C}^\infty(X_G, L_G^p \otimes E_G)$  be the natural identification.

By a result of Zhang [43, Theorem 1.1 and Proposition 1.2], one sees that for  $p$  large enough, the map

$$\pi_G \circ i^* : \mathcal{C}^\infty(X, L^p \otimes E)^G \rightarrow \mathcal{C}^\infty(X_G, L_G^p \otimes E_G)$$

induces a natural isomorphism

$$(0.27) \quad \sigma_p = \pi_G \circ i^* : H^0(X, L^p \otimes E)^G \rightarrow H^0(X_G, L_G^p \otimes E_G).$$

(When  $E = \mathbb{C}$ , this result was first proved in [20, Theorem 3.8].)

The following result is a symplectic version of the above isomorphism which is proved in Corollary 4.10, as a simple application of the Toeplitz operator type properties proved in that subsection. It might be regarded as an “asymptotic symplectic quantization identification”, generalizing the corresponding holomorphic identification (0.27).

**Theorem 0.9.** *If  $X$  is a compact symplectic manifold and  $\mathbf{J} = J$ , then the natural map  $\sigma_p : (\text{Ker } D_p)^G \rightarrow \text{Ker } D_{G,p}$  defined in (4.108) is an isomorphism for  $p$  large enough.*

Let  $\langle \cdot, \cdot \rangle_{L_G^p \otimes E_G}$  be the metric on  $L_G^p \otimes E_G$  induced by  $h^{L_G}$  and  $h^{E_G}$ .

In view of [40, (3.54)], the natural Hermitian product on  $\mathcal{C}^\infty(X_G, L_G^p \otimes E_G)$  is the following weighted Hermitian product  $\langle \cdot, \cdot \rangle_{\tilde{h}}$ :

$$(0.28) \quad \langle s_1, s_2 \rangle_{\tilde{h}} = \int_{X_G} \langle s_1, s_2 \rangle_{L_G^p \otimes E_G}(x_0) \tilde{h}^2(x_0) dv_{X_G}(x_0).$$

In fact,  $\pi_G : (\mathcal{C}^\infty(P, L^p \otimes E)^G, \langle \cdot, \cdot \rangle) \rightarrow (\mathcal{C}^\infty(X_G, L_G^p \otimes E_G), \langle \cdot, \cdot \rangle_{\tilde{h}})$  is an isometry.

We still denote by  $\langle \cdot, \cdot \rangle$  the scalar product on  $H^0(X, L^p \otimes E)^G$  induced by (0.23).

**Theorem 0.10.** *The isomorphism  $(2p)^{-\frac{n_0}{4}} \sigma_p$  is an asymptotic isometry from  $(H^0(X, L^p \otimes E)^G, \langle \cdot, \cdot \rangle)$  onto  $(H^0(X_G, L_G^p \otimes E_G), \langle \cdot, \cdot \rangle_{\tilde{h}})$ , i.e., if  $\{s_i^p\}_{i=1}^{d_p}$  is an orthonormal basis of  $(H^0(X, L^p \otimes E)^G, \langle \cdot, \cdot \rangle)$ , then*

$$(0.29) \quad (2p)^{-\frac{n_0}{2}} \langle \sigma_p s_i^p, \sigma_p s_j^p \rangle_{\tilde{h}} = \delta_{ij} + \mathcal{O}\left(\frac{1}{p}\right).$$

From the explicit formula (0.26), one can also get the coefficient of  $p^{-1}$  in the expansion (0.29). We leave it to the interested readers.

Let  $\tilde{P}_p^{X_G}$  denote the orthogonal projection from  $(\mathcal{C}^\infty(X_G, L_G^p \otimes E_G), \langle \cdot, \cdot \rangle_{\tilde{h}})$  onto  $H^0(X, L_G^p \otimes E_G)$ . Let  $\tilde{P}_p^{X_G}(x_0, x'_0)$  ( $x_0, x'_0 \in X_G$ ) be the smooth kernel of the operator  $\tilde{P}_p^{X_G}$  with respect to  $\tilde{h}^2(x'_0) dv_{X_G}(x'_0)$ .

The following result is an easy consequence of [17, Theorem 1.3].

**Theorem 0.11.** *Under the assumption of Theorem 0.6, there exist smooth coefficients  $\tilde{\Phi}_r(x_0) \in \text{End}(E_G)_{x_0}$  which are polynomials in  $R^{TX_G}, R^{E_G}$  (resp.  $\tilde{h}$ ), and their derivatives at  $x_0$  to order  $2r-1$  (resp.  $2r$ ), and  $\tilde{\Phi}_0 = \text{Id}_{E_G}$ , such that for any  $k, l \in \mathbb{N}$ , there exists  $C_{k,l} > 0$  such that for any  $x_0 \in X_G, p \in \mathbb{N}$ ,*

$$(0.30) \quad \left| p^{-n+n_0} \tilde{h}^2(x_0) \tilde{P}_p^{X_G}(x_0, x_0) - \sum_{r=0}^k \tilde{\Phi}_r(x_0) p^{-r} \right|_{\mathcal{C}^l} \leq C_{k,l} p^{-k-1}.$$

Moreover, the following identity holds,

$$(0.31) \quad \tilde{\Phi}_1(x_0) = \frac{1}{8\pi} r_{x_0}^{X_G} + \frac{1}{2\pi} \Delta_{X_G} \log \tilde{h} + \frac{1}{2\pi} R_{x_0}^{E_G}(w_j^0, \bar{w}_j^0).$$

**Remark 0.12.** From (0.25) and (0.31), one sees that in general  $\Phi_1 \neq \tilde{\Phi}_1$ , if  $\tilde{h}$  is not constant on  $X_G$ . This reflects a subtle difference between the Bergman kernel and the geometric quantization.

From the works [17], [26] and the present paper, we see clearly that the asymptotic expansion of Bergman kernel is parallel to the small time asymptotic expansion of the heat kernel. To localize the problem, the spectral gap property (2.13) and the finite propagation speed of solutions of hyperbolic equations play essential roles.

Let  $U$  be a  $G$ -neighborhood of  $\mu^{-1}(0)$  as in Theorem 0.2, in this paper, we will then work on  $U/G$ .

Indeed, after doing suitable rescaling on the coordinate, we get the limit operator  $\mathcal{L}_2^0$  (cf. (3.13)) which is the sum of two parts, along  $T_{x_0} X_G$ , its kernel is infinite dimensional and gives us the classical Bergman kernel as in  $\mathbb{C}^{n-n_0}$ , while along  $N_G$ , it is a harmonic

oscillator and its kernel is one dimensional. This explains well why we can expect to get the fast decay estimate along  $N_G$  in (0.14).

This paper is organized as follows. In Section 1, we study connections and Laplacians associated to a principal bundle. In Section 2, we localize the problem by using the spectral gap property and finite propagation speed, then we use the rescaling technique in local index theory to prove Theorem 2.23 which is a version of Theorem 0.2 without assumption on  $\mathbf{J}$ . We assume  $G$  acts freely on  $P = \mu^{-1}(0)$  in Sections 2.5-2.8, and in Section 4.1 we explain Theorem 4.1, the version of Theorem 0.2 where we only assume that  $\mu$  is regular at 0. In Section 3, we get explicit informations on the coefficients  $P^{(r)}$  when  $\mathbf{J}$  verifies (3.2), thus we get an effective way to compute its first coefficients of the asymptotic expansion (0.14). Especially, we establish (0.12) and (0.13). In Section 4, we explain various applications of our Theorem 0.2, including Toeplitz properties, etc. In Section 5, we compute the coefficients  $\Phi_1$  in Theorem 0.6 and  $P_{x_0}^{(2)}(0, 0)$  in Theorem 0.7 and in the general case:  $\mathbf{J} \neq J$ . In Section 6, we prove Theorems 0.10, 0.11.

Some results of this paper have been announced in [29, 30].

Notation : In the whole paper, if there is no other specific notification, when in a formula a subscript index appears two times, we sum up with this index.

## 1. CONNECTIONS AND LAPLACIANS ASSOCIATED TO A PRINCIPAL BUNDLE

In this Section, for  $\pi : X \rightarrow B = X/G$  a  $G$ -principal bundle, we will study the associated connections and Bochner-Laplacians. The results in this Section extend the corresponding ones in [2, §1d)] and [1, §5.1, 5.2] where the metric along the fiber is parallel along the horizontal direction. These results will be used in Proposition 2.7 and in Sections 3.3, 5.

If  $G$  acts only infinitesimal freely on  $X$ , then  $B = X/G$  is an orbifold. The results in this Section can be extended easily to this situation, as will be explained in Section 4.1.

This Section is organized as follows. In Section 1.1, we study the Levi-Civita connection for a principal bundle which extends the results of [2, §1d)]. In Section 1.2, we study the relation of the Laplacians on the total and base manifolds.

**1.1. Connections associated to a principal bundle.** Let a compact connected Lie group  $G$  acts smoothly on the left on a smooth manifold  $X$  and  $\dim G = n_0$ . We suppose temporary that  $G$  acts freely on  $X$ . Then

$$\pi : X \rightarrow B = X/G$$

is a  $G$ -principal bundle. We denote by  $TY$  the relative tangent bundle for the fibration  $\pi : X \rightarrow B$ .

Let  $g^{TX}$  be a  $G$ -invariant metric on  $TX$ . Let  $\nabla^{TX}$  be the Levi-Civita connection on  $TX$ . By the explicit equation for  $\langle \nabla^{TX} \cdot, \cdot \rangle$  in [1, (1.18)], for  $W, Z, Z'$  vector fields on  $X$ ,

$$(1.1) \quad 2 \langle \nabla_W^{TX} Z, Z' \rangle = W \langle Z, Z' \rangle + Z \langle W, Z' \rangle - Z' \langle W, Z \rangle \\ - \langle W, [Z, Z'] \rangle - \langle Z, [W, Z'] \rangle + \langle Z', [W, Z] \rangle.$$

Let  $T^H X$  be the orthogonal complement of  $TY$  in  $TX$ .

For  $U \in TB$ , let  $U^H \in T^H X$  be the lift of  $U$ .

Let  $g^{TY}, g^{T^H X}$  be  $G$ -invariant metrics on  $TY, T^H X$  induced by  $g^{TX}$ . Let  $P^{TY}, P^{T^H X}$  be the orthogonal projections from  $TX$  onto  $TY, T^H X$ .

Let  $g^{TB}$  be the metric on  $TB$  induced by  $g^{T^H X}$ . Let  $\nabla^{TB}$  be the Levi-Civita connection on  $(TB, g^{TB})$  with curvature  $R^{TB}$ . Set

$$(1.2) \quad \nabla^{T^H X} = \pi^* \nabla^{TB}, \quad \nabla^{TY} = P^{TY} \nabla^{TX} P^{TY}, \quad {}^0 \nabla^{TX} = \nabla^{TY} \oplus \nabla^{T^H X}.$$

Then  $\nabla^{T^H X}, {}^0 \nabla^{TX}$  define Euclidean connections on  $T^H X, TX$ , and  $\nabla^{TY}$  is the connection on  $TY$  induced by  $\nabla^{TX}$  (cf. [2, Def. 1.6]).

Let  $T$  be the torsion of  ${}^0 \nabla^{TX}$ , and let  $S \in T^* X \otimes \text{End}(TX)$ ,  $\dot{g}^{TY} \in T^* B \otimes \text{End}(TY)$  be defined by

$$(1.3) \quad S = \nabla^{TX} - {}^0 \nabla^{TX}, \quad \dot{g}_U^{TY} = (g^{TY})^{-1} (L_{U^H} g^{TY}) \quad \text{for } U \in TB.$$

Then  $S$  is a 1-form on  $X$  taking values in the skew-adjoint endomorphisms of  $TX$ .

By [6, Theorem 1.2] (cf. [5, Theorems 1.1 and 1.2]) the proof of which can also be found in [1, Prop. 10.2] where one applies directly (1.1), we know that  $\nabla^{TY}$  is the

Levi-Civita connection on  $TY$  along the fiber  $Y$ , and for  $U \in TB$ ,

$$(1.4) \quad \nabla_{U^H}^{TY} = L_{U^H} + \frac{1}{2}(g^{TY})^{-1}(L_{U^H}g^{TY}) = L_{U^H} + \frac{1}{2}\dot{g}_U^{TY}.$$

Let  $\mathfrak{g}$  be the Lie algebra of  $G$ . For  $K \in \mathfrak{g}$ , we denote by  $K_x^X = \frac{\partial}{\partial t}e^{-tK}x|_{t=0}$  the corresponding vector field on  $X$ , then  $gK_x^X = (\text{Ad}_g(K))_{gx}^X$ . Thus we can identify the trivial bundle  $X \times \mathfrak{g}$  with Ad-action of  $G$  on  $\mathfrak{g}$  to the  $G$ -equivariant bundle  $TY$  by the map  $K \rightarrow K^X$ .

Let  $\theta : TX \rightarrow \mathfrak{g}$  be the connection form of the principal bundle  $\pi : X \rightarrow B$  such that  $T^H X = \text{Ker } \theta$ , and  $\Theta$  its curvature.

For  $K_1, K_2 \in \mathfrak{g}$ ,  $U, V \in TB$ , as  $U^H$  is  $G$ -invariant, we have

$$(1.5) \quad L_{U^H}K_1^X = -[K_1^X, U^H] = 0.$$

By (1.4), (1.5), we get  $T \in \Lambda^2(T^*X) \otimes TY$  and

$$(1.6) \quad \begin{aligned} T(U^H, V^H) &= \Theta(U^H, V^H) = -P^{TY}[U^H, V^H], \quad T(K_1^X, K_2^X) = 0, \\ T(U^H, K_1^X) &= \frac{1}{2}(g^{TY})^{-1}(L_{U^H}g^{TY})K_1^X = \frac{1}{2}\dot{g}_U^{TY}K_1^X. \end{aligned}$$

And by (1.1), (1.4), (1.5) and (1.6), for  $W \in TX$ , we have (cf. also [2, (1.28)], [1, Prop. 10.6]),

$$(1.7) \quad \begin{aligned} S(W)(TY) &\subset T^H X, \quad S(U^H)V^H \in TY, \\ 2\langle S(U^H)K_1^X, V^H \rangle &= 2\langle S(K_1^X)U^H, V^H \rangle = \langle T(U^H, V^H), K_1^X \rangle, \\ \langle S(K_2^X)U^H, K_1^X \rangle &= -\langle S(K_2^X)K_1^X, U^H \rangle \\ &= \frac{1}{2}U^H \langle K_2^X, K_1^X \rangle = \langle T(U^H, K_1^X), K_2^X \rangle. \end{aligned}$$

Let  $\{e_i\}$  be an orthonormal basis of  $TB$ . By (1.3) and (1.7), for  $Y$  a section of  $TY$ ,

$$(1.8) \quad \nabla_{U^H}^{TX}Y = \nabla_{U^H}^{TY}Y + \frac{1}{2}\langle T(U^H, e_i^H), Y \rangle e_i^H.$$

**Proposition 1.1.** *Let  $\{f_l\}_{l=1}^{n_0}$  be a  $G$ -invariant orthonormal frame of  $TY$ , then*

$$(1.9) \quad \sum_{l=1}^{n_0} \nabla_{f_l}^{TY} f_l = 0.$$

*Proof.* (1.9) is analogue to the fact that any left invariant volume form on  $G$  is also right invariant. We only need to work on a fiber  $Y_b$ ,  $b \in B$ .

Let  $dv_Y$  be the Riemannian volume form on  $Y_b$ .

By using  $L_{f_k}f_l = \nabla_{f_k}^{TY}f_l - \nabla_{f_l}^{TY}f_k$  and  $dv_Y$  is preserved by  $\nabla^{TY}$  on  $Y_b$ , we get

$$(1.10) \quad L_{f_k}dv_Y = \sum_{l=1}^{n_0} \langle \nabla_{f_l}^{TY} f_k, f_l \rangle dv_Y.$$

Now from  $L_{f_k} = i_{f_k}d^Y + d^Y i_{f_k}$  and  $\langle \nabla_{f_l}^{TY} f_k, f_l \rangle$  is  $G$ -invariant and (1.10), we get

$$(1.11) \quad 0 = \int_{Y_b} L_{f_k}dv_Y = \sum_{l=1}^{n_0} \langle \nabla_{f_l}^{TY} f_k, f_l \rangle \int_{Y_b} dv_Y.$$

From (1.11), we get (1.9).  $\square$

**Remark 1.2.** If  $g^{TY}$  is induced by a family of  $Ad$ -invariant metric on  $\mathfrak{g}$  under the isomorphism from  $X \times \mathfrak{g}$  to  $TY$  defined by  $K \rightarrow K^X$ , then (1.9) is trivial. In this case, as in [19, Theorem 11.3], for  $Y_1, Y_2$  two  $G$ -invariant sections of  $TY$ , by (1.1), we have

$$(1.12) \quad \nabla_{Y_1}^{TY} Y_2 = \frac{1}{2}[Y_1, Y_2].$$

**1.2. Curvatures and Laplacians associated to a principal bundle.** Let  $(F, h^F)$  be a  $G$ -equivariant Hermitian vector bundle on  $X$  with a  $G$ -invariant Hermitian connection  $\nabla^F$  on  $X$ . For any  $K \in \mathfrak{g}$ , denote by  $L_K$  the infinitesimal action induced by  $K$  on the corresponding vector bundles.

Let  $\mu^F$  be the section of  $\mathfrak{g}^* \otimes \text{End}(F)$  on  $X$  defined by,

$$(1.13) \quad \mu^F(K) = \nabla_{K^X}^F - L_K \quad \text{for } K \in \mathfrak{g}.$$

By using the identification  $X \times \mathfrak{g} \rightarrow TY$ ,  $\mu^F$  defines a  $G$ -invariant section  $\tilde{\mu}^F$  of  $TY \otimes \text{End}(F)$  on  $X$  such that

$$(1.14) \quad \langle \tilde{\mu}^F, K^X \rangle = \mu^F(K).$$

The curvature  $R_\mu^F$  of the Hermitian connection  $\nabla^F - \mu^F(\theta)$  on  $F$  is  $G$ -invariant. Moreover as  $\nabla^F$  is  $G$ -invariant, by (1.13),

$$(1.15) \quad R_\mu^F(K^X, v) = [L_K, \nabla^F - \mu^F(\theta)](v) = 0$$

for  $K \in \mathfrak{g}$ ,  $v \in TX$ , and

$$(1.16) \quad R_\mu^F = R^F - \nabla^F(\mu^F(\theta)) + \mu^F(\theta) \wedge \mu^F(\theta).$$

The Hermitian vector bundle  $(F, h^F)$  induces a Hermitian vector bundle  $(F_B, h^{F_B})$  on  $B$  by identifying  $G$ -invariant sections of  $F$  on  $X$ .

For  $s \in \mathcal{C}^\infty(B, F_B) \simeq \mathcal{C}^\infty(X, F)^G$ , we define

$$(1.17) \quad \nabla_U^{F_B} s = \nabla_{U^H}^F s.$$

Then  $\nabla^{F_B}$  is a Hermitian connection on  $F_B$  with curvature  $R^{F_B}$ .

Observe that  $\nabla^{F_B}$  is the restriction of the connection  $\nabla^F - \mu^F(\theta)$  to  $\mathcal{C}^\infty(X, F)^G$ , and  $R^{F_B}$  is the section induced by  $R_\mu^F$ . From (1.16), for  $U_1, U_2 \in TB$ , we get

$$(1.18) \quad R^{F_B}(U_1, U_2) = R^F(U_1^H, U_2^H) - \mu^F(\Theta)(U_1, U_2).$$

Let  $dv_X$  be the Riemannian volume form on  $(X, g^{TX})$ . We define a scalar product on  $\mathcal{C}^\infty(X, F)$  by

$$(1.19) \quad \langle s_1, s_2 \rangle = \int_X \langle s_1, s_2 \rangle_F(x) dv_X(x).$$

As in (1.19),  $h^{F_B}$ ,  $g^{TB}$  induce a natural scalar product  $\langle \cdot, \cdot \rangle$  on  $\mathcal{C}^\infty(B, F_B)$ .

Denote by  $\text{vol}(Gx)$  ( $x \in X$ ) the volume of the orbit  $Gx$  equipped with the metric induced by  $g^{TX}$ . The function

$$h(x) = \sqrt{\text{vol}(Gx)}, \quad x \in X,$$



as in (0.10) is  $G$ -invariant and defines a function on  $B$ .

Denote by  $\pi_G : \mathcal{C}^\infty(X, F)^G \rightarrow \mathcal{C}^\infty(B, F_B)$  the natural identification. Then the map

$$(1.20) \quad \Phi = h\pi_G : (\mathcal{C}^\infty(X, F)^G, \langle \cdot, \cdot \rangle) \rightarrow (\mathcal{C}^\infty(B, F_B), \langle \cdot, \cdot \rangle)$$

is an isometry.

Let  $\{e_a\}_{a=1}^m$  be an orthonormal frame of  $TX$ .

Let  $(E, h^E)$  be a Hermitian vector bundle on  $X$  and let  $\nabla^E$  be a Hermitian connection on  $E$ . The usual Bochner Laplacians  $\Delta^E, \Delta_X$  are defined by

$$(1.21) \quad \Delta^E := - \sum_{a=1}^m \left( (\nabla_{e_a}^E)^2 - \nabla_{\nabla_{e_a}^{TX} e_a}^E \right), \quad \Delta_X = \Delta^{\mathbb{C}}.$$

Let  $\{f_l\}_{l=1}^{n_0}$  be a  $G$ -invariant orthonormal frame of  $TY$ , and  $\{f^l\}$  its dual basis, and let  $\{e_i\}$  be an orthonormal frame of  $TB$ , then  $\{e_i^H, f_l\}$  is an orthonormal frame of  $TX$ .

To simplify the notation, for  $\sigma_1, \sigma_2 \in TY \otimes \text{End}(F)$ , we denote by  $\langle \sigma_1, \sigma_2 \rangle_{g^{TY}} \in \text{End}(F)$  the contraction of  $\sigma_1 \otimes \sigma_2$  on the part of  $TY$  by  $g^{TY}$ . In particular,

$$(1.22) \quad \langle \tilde{\mu}^F, \tilde{\mu}^F \rangle_{g^{TY}} = \sum_{l=1}^{n_0} \langle \tilde{\mu}^F, f_l \rangle^2 \in \text{End}(F).$$

The following result extends [1, Prop. 5.6, 5.10] where  $F = X \times_G V$  for a  $G$ -representation  $V$ , and where  $g^{TY}$  is induced by a fixed  $Ad$ -invariant metric on  $\mathfrak{g}$  under the isomorphism from  $X \times \mathfrak{g}$  to  $TY$  defined by  $K \rightarrow K^X$  (Thus  $h$  is constant on  $B$ ).

**Theorem 1.3.** *As an operator on  $\mathcal{C}^\infty(B, F_B)$ , we have*

$$(1.23) \quad \Phi \Delta^F \Phi^{-1} = \Delta^{F_B} - \langle \tilde{\mu}^F, \tilde{\mu}^F \rangle_{g^{TY}} - \frac{1}{h} \Delta_B h.$$

*Proof.* At first by (1.6) and (1.7),

$$(1.24) \quad \begin{aligned} \frac{1}{h}(e_i h) &= \frac{1}{2}(L_{e_i^H} dv_Y)/dv_Y = \frac{1}{2} \langle L_{e_i^H} f^l, f^l \rangle = -\frac{1}{2} \langle L_{e_i^H} f_l, f_l \rangle \\ &= \frac{1}{4}(L_{e_i^H} g^{TY})(f_l, f_l) = \frac{1}{2} \langle T(e_i^H, f_l), f_l \rangle = -\frac{1}{2} \langle S(f_l) f_l, e_i^H \rangle. \end{aligned}$$

As  $\tilde{\mu}^F$  is  $G$ -invariant, then  $\langle \tilde{\mu}^F, f_l \rangle$  is also a  $G$ -invariant section of  $\text{End}(F)$ .

By (1.13),  $\nabla_{f_l}^F = \langle \tilde{\mu}^F, f_l \rangle$  on  $\mathcal{C}^\infty(X, F)^G$ , and by (1.3),  $\nabla_{f_l}^{TX} f_l = \nabla_{f_l}^{TY} f_l + S(f_l) f_l$ , thus by (1.20), we get for  $1 \leq l \leq n_0$ ,

$$(1.25) \quad \Phi[(\nabla_{f_l}^F)^2 - \nabla_{\nabla_{f_l}^{TX} f_l}^F] \Phi^{-1} = \langle \tilde{\mu}^F, f_l \rangle^2 - \langle \tilde{\mu}^F, \nabla_{f_l}^{TY} f_l \rangle - h \nabla_{S(f_l) f_l}^{F_B} h^{-1}.$$

From (1.7), (1.9), (1.21), (1.22), (1.24) and (1.25), we have

$$(1.26) \quad \begin{aligned} \Phi \Delta^F \Phi^{-1} &= - \sum_{i=1}^{2n-n_0} \Phi \left[ (\nabla_{e_i^H}^F)^2 - \nabla_{\nabla_{e_i^H}^{TX} e_i^H}^F \right] \Phi^{-1} - \sum_{l=1}^{n_0} \Phi \left[ (\nabla_{f_l}^F)^2 - \nabla_{\nabla_{f_l}^{TX} f_l}^F \right] \Phi^{-1} \\ &= h \Delta^{F_B} h^{-1} - \sum_{l=1}^{n_0} \langle \tilde{\mu}^F, f_l \rangle^2 - 2(e_i h) \nabla_{e_i^H}^{F_B} h^{-1} = \Delta^{F_B} - \langle \tilde{\mu}^F, \tilde{\mu}^F \rangle_{g^{TY}} - \frac{1}{h} \Delta_B h. \end{aligned}$$

□

## 2. $G$ -INVARIANT BERGMAN KERNELS

In this Section, we study the uniform estimate with its derivatives on  $t = \frac{1}{\sqrt{p}}$  of the  $G$ -invariant Bergman kernel  $P_p^G(x, x')$  of  $D_p^2$  as  $p \rightarrow \infty$ .

The first main difficulty is to localize the problem to arbitrary small neighborhoods of  $P = \mu^{-1}(0)$ , so that one can study the  $G$ -invariant Bergman kernel in the spirit of [17]. Our observation here is that the  $G$ -invariant Bergman kernel is exactly the kernel of the orthogonal projection on the zero space of an operator  $\mathcal{L}_p$ , which is a deformation of  $D_p^2$  by the Casimir operator. Moreover,  $\mathcal{L}_p$  has a spectral gap property (cf. (2.22), (2.23)). In the spirit of [17, §3], this allows us to localize the problem to a problem near a  $G$ -neighborhood of  $Gx$ . By combining with the Lichnerowicz formula, we get Theorem 0.1 in Section 2.4.

After localizing the problem to a problem near  $P$ , we first replace  $X$  by  $G \times \mathbb{R}^{2n-n_0}$ , then we reduce it to a problem on  $\mathbb{R}^{2n-n_0}$ . On  $\mathbb{R}^{2n-n_0}$ , the problem in Section 2.7 is similar to a problem on  $\mathbb{R}^{2n}$  considered in [17, §3.3].

Comparing with the operator in [17, §3.3], we have an extra quadratic term along the normal direction of  $X_G$ . This allows us to improve the estimate in the normal direction. After suitable rescaling, we will introduce a family of Sobolev norms defined by the rescaled connection on  $L^p$  and the rescaled moment map in this situation, then we can extend the functional analysis techniques developed in [17, §3.3] and [7, §11].

This section is organized as follows. In Section 2.1, we recall a basic property on the Casimir operator of a compact connected Lie group. In Section 2.2, we recall the definition of  $\text{spin}^c$  Dirac operators for an almost complex manifold. In Section 2.3, we introduce the operator  $\mathcal{L}_p$  to study the  $G$ -invariant Bergman kernel  $P_p^G$  of  $D_p^2$ . In Section 2.4, we explain that the asymptotic expansion of  $P_p^G(x, x')$  is localized on a  $G$ -neighborhood of  $Gx$ , and we establish Theorem 0.1. In Section 2.5, we show that our problem near  $P$  is equivalent to a problem on  $U/G$  for any open  $G$ -neighborhood  $U$  of  $P$ . In Section 2.6, we derive an asymptotic expansion of  $\Phi \mathcal{L}_p \Phi^{-1}$  in coordinates of  $U/G$ . In Section 2.7, we study the uniform estimate with its derivatives on  $t$  of the Bergman kernel associated to the rescaled operator  $\mathcal{L}_2^t$  from  $\Phi \mathcal{L}_p \Phi^{-1}$  using heat kernel. In Theorem 2.21, we estimate uniformly the remainder term of the Taylor expansion of  $e^{-u\mathcal{L}_2^t}$  for  $u \geq u_0 > 0$ ,  $0 < t \leq t_0 \leq 1$ . In Section 2.8, we identify  $J_{r,u}$ , the coefficient of the Taylor expansion of  $e^{-u\mathcal{L}_2^t}$ , with the Volterra expansion of the heat kernel, thus giving a way to compute the coefficient  $P_{x_0}^{(r)}$  in Theorem 0.2. In Section 2.8, we prove Theorem 0.2 except (0.12) and (0.13).

We use the notation in Section 1. In Sections 2.5-2.8, we assume  $G$  acts freely on  $P = \mu^{-1}(0)$ .

**2.1. Casimir operator.** Let  $G$  be a compact connected Lie group with Lie algebra  $\mathfrak{g}$  and  $\dim G = n_0$ . We choose an Ad-invariant metric on  $\mathfrak{g}$  such that it is the minus Killing form on the semi-simple part of  $\mathfrak{g}$ .

Let  $\{K_j\}_{j=1}^{\dim G}$  be an orthogonal basis of  $\mathfrak{g}$  and  $\{K^j\}$  be its dual basis of  $\mathfrak{g}^*$ .

The Casimir operator  $\text{Cas}$  of  $\mathfrak{g}$  is defined as the following element of the universal enveloping algebra  $U(\mathfrak{g})$  of  $\mathfrak{g}$ ,

$$(2.1) \quad \text{Cas} := \sum_{j=1}^{\dim G} K_j K_j.$$

Then  $\text{Cas}$  is independent of the choice of  $\{K_j\}$  and belongs to the center of  $U(\mathfrak{g})$ .

Let  $\mathfrak{t}$  be the Lie algebra of a maximum torus  $T$  of  $G$ , and  $\mathfrak{t}^*$  its dual. Let  $|\cdot|$  denote the norm on  $\mathfrak{t}^*$  induced by the Ad-invariant metric on  $\mathfrak{g}$ .

Let  $\mathcal{W} \subset \mathfrak{t}^*$  be the fundamental Weyl chamber associated to the set of positive roots  $\Delta^+$  of  $G$ , and its closure  $\overline{\mathcal{W}} \subset \mathfrak{t}^*$ .

Let  $I = \{K \in \mathfrak{t}; \exp(2\pi K) = 1 \in T\}$  the integer lattice such that  $T = \mathfrak{t}/2\pi I$ , and  $P = \{\alpha \in \mathfrak{t}^*; \alpha(I) \subset \mathbb{Z}\}$  the lattice of integral forms.

Let  $\varrho_G$  be the half sum of the positive roots of  $G$ .

By the Weyl character formula [19, Theorem 8.21], the irreducible representations correspond one to one to  $\vartheta \in \overline{\mathcal{W}} \cap P$ , the highest weight of the representation.

Moreover, for any irreducible representation  $\rho : G \rightarrow \text{End}(E)$  with highest weight  $\vartheta \in \overline{\mathcal{W}} \cap P$ , classically, the action of  $\text{Cas}$  on  $E$  is given by (cf. [19, Theorem 10.6]),

$$(2.2) \quad \rho(\text{Cas}) = -(|\vartheta + \varrho_G|^2 - |\varrho_G|^2) \text{Id}_E.$$

Set

$$(2.3) \quad \nu_1 := \inf_{0 \neq \vartheta \in \overline{\mathcal{W}} \cap P} (|\vartheta + \varrho_G|^2 - |\varrho_G|^2) > 0.$$

By (2.2), for any representation  $\rho : G \rightarrow \text{End}(E)$ , if the  $G$ -invariant subspace  $E^G$  of  $E$  is zero, then

$$(2.4) \quad -\rho(\text{Cas}) \geq \nu_1 \text{Id}_E.$$

**2.2.  $\text{Spin}^c$  Dirac operator.** Let  $(X, \omega)$  be a compact symplectic manifold of real dimension  $2n$ . Assume that there exists a Hermitian line bundle  $L$  over  $X$  endowed with a Hermitian connection  $\nabla^L$  with the property that

$$\frac{\sqrt{-1}}{2\pi} R^L = \omega,$$

where  $R^L = (\nabla^L)^2$  is the curvature of  $(L, \nabla^L)$ .

Let  $(E, h^E)$  be a Hermitian vector bundle on  $X$  with Hermitian connection  $\nabla^E$  and its curvature  $R^E$ .

Let  $g^{TX}$  be a Riemannian metric on  $X$ .

Let  $\mathbf{J} : TX \rightarrow TX$  be the skew-adjoint linear map which satisfies the relation

$$(2.5) \quad \omega(u, v) = g^{TX}(\mathbf{J}u, v)$$

for  $u, v \in TX$ .

Let  $J$  be an almost complex structure such that

$$(2.6) \quad g^{TX}(Ju, Jv) = g^{TX}(u, v), \quad \omega(Ju, Jv) = \omega(u, v),$$

and that  $\omega(\cdot, J\cdot)$  defines a metric on  $TX$ . Then  $J$  commutes with  $\mathbf{J}$  and

$$-\langle J\mathbf{J}\cdot, \cdot \rangle = \omega(\cdot, J\cdot)$$

is positive by our assumption. Thus  $-J\mathbf{J} \in \text{End}(TX)$  is symmetric and positive, and one verifies easily that

$$(2.7) \quad -J\mathbf{J} = (-\mathbf{J}^2)^{1/2}, \quad J = \mathbf{J}(-\mathbf{J}^2)^{-1/2}.$$

The almost complex structure  $J$  induces a splitting

$$TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X,$$

where  $T^{(1,0)}X$  and  $T^{(0,1)}X$  are the eigenbundles of  $J$  corresponding to the eigenvalues  $\sqrt{-1}$  and  $-\sqrt{-1}$  respectively. Let  $T^{*(1,0)}X$  and  $T^{*(0,1)}X$  be the corresponding dual bundles.

For any  $v \in TX \otimes_{\mathbb{R}} \mathbb{C}$  with decomposition  $v = v_{1,0} + v_{0,1} \in T^{(1,0)}X \oplus T^{(0,1)}X$ , let  $\bar{v}_{1,0}^* \in T^{*(0,1)}X$  be the metric dual of  $v_{1,0}$ . Then

$$(2.8) \quad c(v) := \sqrt{2}(\bar{v}_{1,0}^* \wedge -i_{v_{0,1}})$$

defines the Clifford action of  $v$  on  $\Lambda(T^{*(0,1)}X)$ , where  $\wedge$  and  $i$  denote the exterior and interior multiplications respectively.

Set

$$(2.9) \quad \nu_0 := \inf_{u \in T_x^{(1,0)}X, x \in X} R_x^L(u, \bar{u})/|u|_{g^{TX}}^2 > 0.$$

Let  $\nabla^{TX}$  be the Levi-Civita connection of the metric  $g^{TX}$  with curvature  $R^{TX}$ . We denote by  $P^{T^{(1,0)}X}$  the projection from  $TX \otimes_{\mathbb{R}} \mathbb{C}$  to  $T^{(1,0)}X$ .

Let  $\nabla^{T^{(1,0)}X} = P^{T^{(1,0)}X} \nabla^{TX} P^{T^{(1,0)}X}$  be the Hermitian connection on  $T^{(1,0)}X$  induced by  $\nabla^{TX}$  with curvature  $R^{T^{(1,0)}X}$ .

By [22, pp.397–398],  $\nabla^{TX}$  induces canonically a Clifford connection  $\nabla^{\text{Cliff}}$  on  $\Lambda(T^{*(0,1)}X)$  and its curvature  $R^{\text{Cliff}}$  (cf. also [25, §2]).

Let  $\{e_a\}_a$  be an orthonormal basis of  $TX$ . Then

$$(2.10) \quad R^{\text{Cliff}} = \frac{1}{4} \sum_{ab} \langle R^{TX} e_a, e_b \rangle c(e_a) c(e_b) + \frac{1}{2} \text{Tr} [R^{T^{(1,0)}X}].$$

Let  $\nabla^{E_p}$  be the connection on

$$(2.11) \quad E_p := \Lambda(T^{*(0,1)}X) \otimes L^p \otimes E$$

induced by  $\nabla^{\text{Cliff}}$ ,  $\nabla^L$  and  $\nabla^E$ .

Let  $\langle \cdot \rangle_{E_p}$  be the metric on  $E_p$  induced by  $g^{TX}$ ,  $h^L$  and  $h^E$ .

The  $L^2$ -scalar product  $\langle \cdot \rangle$  on  $\Omega^{0,\bullet}(X, L^p \otimes E)$ , the space of smooth sections of  $E_p$ , is given by (1.19). We denote the corresponding norm by  $\|\cdot\|_{L^2}$ .

**Definition 2.1.** The spin<sup>c</sup> Dirac operator  $D_p$  is defined by

$$(2.12) \quad D_p := \sum_{a=1}^{2n} c(e_a) \nabla_{e_a}^{E_p} : \Omega^{0,\bullet}(X, L^p \otimes E) \longrightarrow \Omega^{0,\bullet}(X, L^p \otimes E).$$

Clearly,  $D_p$  is a formally self-adjoint, first order elliptic differential operator on  $\Omega^{0,\bullet}(X, L^p \otimes E)$ , which interchanges  $\Omega^{0,\text{even}}(X, L^p \otimes E)$  and  $\Omega^{0,\text{odd}}(X, L^p \otimes E)$ .

If  $A$  is any operator, we denote by  $\text{Spec}(A)$  the spectrum of  $A$ .

The following result was proved in [25, Theorems 1.1, 2.5] by applying directly the Lichnerowicz formula (cf. also [8, Theorem 1] in the holomorphic case).

**Theorem 2.2.** *There exists  $C_L > 0$  such that for any  $p \in \mathbb{N}$  and any  $s \in \Omega^{>0}(X, L^p \otimes E) = \bigoplus_{q \geq 1} \Omega^{0,q}(X, L^p \otimes E)$ ,*

$$(2.13) \quad \|D_p s\|_{L^2}^2 \geq (2p\nu_0 - C_L) \|s\|_{L^2}^2.$$

Moreover  $\text{Spec}(D_p^2) \subset \{0\} \cup [2p\nu_0 - C_L, +\infty[$ .

**2.3.  $G$ -invariant Bergman kernel.** Suppose that the compact connected Lie group  $G$  acts on the left of  $X$ , and the action of  $G$  lifts on  $L, E$  and preserves the metrics and connections,  $\omega$  and the almost complex structure  $J$ .

Let  $\mu : X \rightarrow \mathfrak{g}^*$  be defined by

$$(2.14) \quad 2\pi\sqrt{-1}\mu(K) := \mu^L(K) = \nabla_{K^X}^L - L_K, \quad K \in \mathfrak{g}.$$

Then  $\mu$  is the corresponding moment map (cf. [1, Example. 7.9]), i.e. for any  $K \in \mathfrak{g}$ ,

$$(2.15) \quad d\mu(K) = i_{K^X}\omega.$$

For  $V$  a subspace of  $\Omega^{0,\bullet}(X, L^p \otimes E)$ , we denote by  $V^\perp$  the orthogonal complement of  $V$  in  $(\Omega^{0,\bullet}(X, L^p \otimes E), \langle \cdot, \cdot \rangle)$ .

Let  $\Omega^{0,\bullet}(X, L^p \otimes E)^G$ ,  $(\text{Ker } D_p)^G$  be the  $G$ -invariant subspaces of  $\Omega^{0,\bullet}(X, L^p \otimes E)$ ,  $\text{Ker } D_p$ . Let  $P_p^G$  be the orthogonal projection from  $\Omega^{0,\bullet}(X, L^p \otimes E)$  on  $(\text{Ker } D_p)^G$ .

**Definition 2.3.** The  $G$ -invariant Bergman kernel  $P_p^G(x, x')$  ( $x, x' \in X$ ) of  $D_p$  is the smooth kernel of  $P_p^G$  with respect to the Riemannian volume form  $dv_X(x')$ .

Let  $\{S_i^p\}_{i=1}^{d_p}$  ( $d_p = \dim(\text{Ker } D_p)^G$ ) be any orthonormal basis of  $(\text{Ker } D_p)^G$  with respect to the norm  $\|\cdot\|_{L^2}$ , then

$$(2.16) \quad P_p^G(x, x') = \sum_{i=1}^{d_p} S_i^p(x) \otimes (S_i^p(x'))^* \in (E_p)_x \otimes (E_p^*)_{x'}.$$

Especially,  $P_p^G(x, x) \in \text{End}(E_p)_x \simeq \text{End}(\Lambda(T^{*(0,1)}X) \otimes E)_x$ .

We use the notation  $\mu^F$  in (1.13) now.

Recall that the Lie derivative  $L_K$  on  $TX$  is given by

$$(2.17) \quad L_K V = \nabla_{K^X}^{TX} V - \nabla_V^{TX} K^X.$$

Thus

$$(2.18) \quad \mu^{TX}(K) = \nabla_{\cdot}^{TX} K^X \in \text{End}(TX),$$

and the action on  $\Lambda(T^{*(0,1)}X)$  induced by  $\mu^{TX}(K)$  is given by

$$(2.19) \quad \mu^{\text{Cliff}}(K) = \frac{1}{4} \sum_{a=1}^{2n} c(e_a) c(\nabla_{e_a}^{TX} K^X) + \frac{1}{2} \text{Tr}[P^{T^{(1,0)}X} \nabla_{\cdot}^{TX} K^X].$$

Thus the action  $L_K$  of  $K$  on smooth sections of  $\Lambda(T^{*(0,1)}X)$  is given by (cf. [40, (1.24)])

$$(2.20) \quad L_K = \nabla_{K^X}^{\text{Cliff}} - \mu^{\text{Cliff}}(K).$$

By (2.14) and (2.20), the action  $L_K$  of  $K$  on  $\Omega^{0,\bullet}(X, L^p \otimes E)$  is  $\nabla_{K^X}^{E_p} - \mu^{E_p}(K)$  with

$$(2.21) \quad \mu^{E_p}(K) = 2\pi\sqrt{-1}p\mu(K) + \mu^E(K) + \mu^{\text{Cliff}}(K).$$

**Definition 2.4.** The (formally) self-adjoint operator  $\mathcal{L}_p$  acting on  $(\Omega^{0,\bullet}(X, L^p \otimes E), \langle \cdot, \cdot \rangle)$  is defined by,

$$(2.22) \quad \mathcal{L}_p = D_p^2 - p \sum_{i=1}^{\dim G} L_{K_i} L_{K_i}.$$

The following result will play a crucial role in the whole paper.

**Theorem 2.5.** *The projection  $P_p^G$  is the orthogonal projection from  $\Omega^{0,\bullet}(X, L^p \otimes E)$  onto  $\text{Ker}(\mathcal{L}_p)$ . Moreover, there exist  $\nu, C_L > 0$  such that for any  $p \in \mathbb{N}$ ,*

$$(2.23) \quad \begin{aligned} \text{Ker}(\mathcal{L}_p) &= (\text{Ker } D_p)^G, \\ \text{Spec}(\mathcal{L}_p) &\subset \{0\} \cup [2p\nu - C_L, +\infty[. \end{aligned}$$

*Proof.* By (2.22), for any  $s \in \Omega^{0,\bullet}(X, L^p \otimes E)$ ,

$$(2.24) \quad \langle \mathcal{L}_p s, s \rangle = \|D_p s\|_{L^2}^2 + p \sum_{i=1}^{\dim G} \|L_{K_i} s\|_{L^2}^2.$$

Thus  $\mathcal{L}_p s = 0$  is equivalent to

$$(2.25) \quad D_p s = L_{K_i} s = 0.$$

This means  $s$  is fixed by the  $G$ -action. Thus we get the first equation of (2.23).

For  $s \in (\text{Ker } \mathcal{L}_p)^\perp$ , there exist  $s_1 \in \Omega^{0,\bullet}(X, L^p \otimes E)^G \cap (\text{Ker } D_p)^\perp$ ,  $s_2 \in (\Omega^{0,\bullet}(X, L^p \otimes E)^G)^\perp$ , such that  $s = s_1 + s_2$ . Clearly,

$$D_p s_1 \in \Omega^{0,\bullet}(X, L^p \otimes E)^G, \quad D_p s_2 \in (\Omega^{0,\bullet}(X, L^p \otimes E)^G)^\perp.$$

By Theorem 2.2 and (2.4),

$$(2.26) \quad \begin{aligned} \langle \mathcal{L}_p s, s \rangle &= -p \langle \rho(\text{Cas}) s_2, s_2 \rangle + \|D_p s_2\|_{L^2}^2 + \|D_p s_1\|_{L^2}^2 \\ &\geq p\nu_1 \|s_2\|_{L^2}^2 + (2p\nu_0 - C_L) \|s_1\|_{L^2}^2, \end{aligned}$$

from which we get (2.23).  $\square$

We assume that  $0 \in \mathfrak{g}^*$  is a regular value of  $\mu$ . Then  $X_G = \mu^{-1}(0)/G$  is an orbifold ( $X_G$  is smooth if  $G$  acts freely on  $P = \mu^{-1}(0)$ ). Furthermore,  $\omega$  descends to a symplectic form  $\omega_G$  on  $X_G$ . Thus one gets the Marsden-Weinstein symplectic reduction  $(X_G, \omega_G)$ .

Moreover,  $(L, \nabla^L)$ ,  $(E, \nabla^E)$  descend to  $(L_G, \nabla^{L_G})$ ,  $(E_G, \nabla^{E_G})$  over  $X_G$  so that the corresponding curvature condition holds [20] :

$$(2.27) \quad \frac{\sqrt{-1}}{2\pi} R^{L_G} = \omega_G.$$

The  $G$ -invariant almost complex structure  $J$  also descends to an almost complex structure  $J_G$  on  $TX_G$ , and  $h^L, h^E, g^{TX}$  descend to  $h^{L_G}, h^{E_G}, g^{TX_G}$ .

We can construct the corresponding  $\text{spin}^c$  Dirac operator  $D_{G,p}$  on  $X_G$ .

Let  $P_{G,p}$  be the orthogonal projection from  $\Omega^{0,\bullet}(X_G, L_G^p \otimes E_G)$  on  $\text{Ker } D_{G,p}$ , and let  $P_{G,p}(x, x')$  be the smooth kernel of  $P_{G,p}$  with respect to the Riemannian volume form  $dv_{X_G}(x')$ .

The purpose of this paper is to study the asymptotic expansion of  $P_p^G(x, x')$  when  $p \rightarrow \infty$ , and we will relate it to the asymptotic expansion of the Bergman kernel  $P_{G,p}$  on  $X_G$ .

**2.4. Localization of the problem.** Let  $a^X$  be the injectivity radius of  $(X, g^{TX})$ , and  $\varepsilon \in (0, a^X/4)$ . If  $x \in X$ ,  $Z \in T_x X$ , let  $\mathbb{R} \ni u \rightarrow x_u = \exp_x^X(uZ) \in X$  be the geodesic in  $(X, g^{TX})$ , such that  $x_0 = x$ ,  $\frac{dx_u}{du}|_{u=0} = Z$ .

For  $x \in X$ , we denote by  $B^X(x, \varepsilon)$  and  $B^{T_x X}(0, \varepsilon)$  the open balls in  $X$  and  $T_x X$  with center  $x$  and radius  $\varepsilon$ , respectively. The map  $T_x X \ni Z \rightarrow \exp_x^X(Z) \in X$  is a diffeomorphism from  $B^{T_x X}(0, \varepsilon)$  on  $B^X(x, \varepsilon)$  for  $\varepsilon \leq a^X$ .

From now on, we identify  $B^{T_x X}(0, \varepsilon)$  with  $B^X(x, \varepsilon)$  for  $\varepsilon \leq a^X/4$ .

Let  $f : \mathbb{R} \rightarrow [0, 1]$  be a smooth even function such that

$$(2.28) \quad f(v) = \begin{cases} 1 & \text{for } |v| \leq \varepsilon/2, \\ 0 & \text{for } |v| \geq \varepsilon. \end{cases}$$

Set

$$(2.29) \quad F(a) = \left( \int_{-\infty}^{+\infty} f(v) dv \right)^{-1} \int_{-\infty}^{+\infty} e^{iva} f(v) dv.$$

Then  $F(a)$  is an even function and lies in Schwartz space  $\mathcal{S}(\mathbb{R})$  and  $F(0) = 1$ .

Let  $\tilde{F}$  be the holomorphic function on  $\mathbb{C}$  such that  $\tilde{F}(a^2) = F(a)$ . The restriction of  $\tilde{F}$  to  $\mathbb{R}$  lies in the Schwartz space  $\mathcal{S}(\mathbb{R})$ .

Let  $\tilde{F}(\mathcal{L}_p)(x, x')$  be the smooth kernel of  $\tilde{F}(\mathcal{L}_p)$  with respect to the volume form  $dv_X(x')$ .

**Proposition 2.6.** *For any  $l, m \in \mathbb{N}$ , there exists  $C_{l,m} > 0$  such that for  $p \geq C_L/\nu$ ,*

$$(2.30) \quad |\tilde{F}(\mathcal{L}_p)(x, x') - P_p^G(x, x')|_{\mathcal{C}^m(X \times X)} \leq C_{l,m} p^{-l}.$$

Here the  $\mathcal{C}^m$  norm is induced by  $\nabla^L, \nabla^E, \nabla^{\text{Cliff}}, h^L, h^E$  and  $g^{TX}$ .

*Proof.* For  $a \in \mathbb{R}$ , set

$$(2.31) \quad \phi_p(a) = 1_{[p\nu, +\infty[}(a) \tilde{F}(a).$$

Then by Theorem 2.5, for  $p > C_L/\nu$ ,

$$(2.32) \quad \tilde{F}(\mathcal{L}_p) - P_p^G = \phi_p(\mathcal{L}_p).$$

By (2.29), for any  $m \in \mathbb{N}$  there exists  $C_m > 0$  such that

$$(2.33) \quad \sup_{a \in \mathbb{R}} |a|^m |\tilde{F}(a)| \leq C_m.$$

As  $X$  is compact, there exist  $\{x_i\}_{i=1}^r \subset X$  such that  $\{U_i = B^X(x_i, \varepsilon)\}_{i=1}^r$  is a covering of  $X$ . We identify  $B^{T_{x_i}X}(0, \varepsilon)$  with  $B^X(x_i, \varepsilon)$  by geodesics as above.

We identify  $(E_p)_Z$  for  $Z \in B^{T_{x_i}X}(0, \varepsilon)$  to  $(E_p)_{x_i}$  by parallel transport with respect to the connection  $\nabla^{E_p}$  along the curve  $\gamma_Z : [0, 1] \ni u \rightarrow \exp_{x_i}^X(uZ)$ .

Let  $\{e_j\}_{j=1}^{2n}$  be an orthonormal basis of  $T_{x_i}X$ . Let  $\tilde{e}_j(Z)$  be the parallel transport of  $e_j$  with respect to  $\nabla^{TX}$  along the above curve.

Let  $\Gamma^E, \Gamma^L, \Gamma^{\text{Cliff}}$  be the corresponding connection forms of  $\nabla^E, \nabla^L$  and  $\nabla^{\text{Cliff}}$  with respect to any fixed frame for  $E, L, \Lambda(T^{*(0,1)}X)$  which is parallel along the curve  $\gamma_Z$  under the trivialization on  $U_i$ . Then  $\Gamma^L$  is a usual 1-form.

Denote by  $\nabla_U$  the ordinary differentiation operator on  $T_{x_i}X$  in the direction  $U$ . Then

$$(2.34) \quad \nabla^{E_p} = \nabla + p\Gamma^L + \Gamma^{\text{Cliff}} + \Gamma^E, \quad D_p = c(\tilde{e}_j)\nabla_{\tilde{e}_j}^{E_p}.$$

Let  $\{\varphi_i\}$  be a partition of unity subordinate to  $\{U_i\}$ .

For  $l \in \mathbb{N}$ , we define a Sobolev norm on the  $l$ -th Sobolev space  $H^l(X, E_p)$  by

$$(2.35) \quad \|s\|_{H_p^l}^2 = \sum_i \sum_{k=0}^l \sum_{i_1, \dots, i_k=1}^{2n} \|\nabla_{e_{i_1}} \cdots \nabla_{e_{i_k}}(\varphi_i s)\|_{L^2}^2$$

Then by (2.34), there exist  $C, C', C'' > 0$  such that for  $p \geq 1, s \in H^2(X, E_p)$ ,

$$(2.36) \quad C'\|D_p^2 s\|_{L^2} - C''p^2\|s\|_{L^2} \leq \|s\|_{H_p^2} \leq C(\|D_p^2 s\|_{L^2} + p^2\|s\|_{L^2}).$$

Observe that  $D_p$  commutes with the  $G$ -action, thus

$$(2.37) \quad [D_p, L_{K_j}] = 0.$$

By (2.22), (2.37), and the facts that  $D_p$  is self-adjoint and  $L_{K_j}$  is skew-adjoint, we know

$$(2.38) \quad \begin{aligned} \|\mathcal{L}_p s\|_{L^2}^2 &= \|D_p^2 s\|_{L^2}^2 + p^2 \left\| \sum_j L_{K_j} L_{K_j} s \right\|_{L^2}^2 - 2p \operatorname{Re} \sum_j \langle D_p^2 s, L_{K_j} L_{K_j} s \rangle \\ &= \|D_p^2 s\|_{L^2}^2 + p^2 \left\| \sum_j L_{K_j} L_{K_j} s \right\|_{L^2}^2 + 2p \sum_j \|L_{K_j} D_p s\|_{L^2}^2. \end{aligned}$$

From (2.36), and (2.38), there exists  $C > 0$  such that

$$(2.39) \quad \|s\|_{H_p^2} \leq C(\|\mathcal{L}_p s\|_{L^2} + p^2\|s\|_{L^2}).$$

Let  $Q$  be a differential operator of order  $m \in \mathbb{N}$  with scalar principal symbol and with compact support in  $U_i$ , then

$$(2.40) \quad [\mathcal{L}_p, Q] = [D_p^2, Q] - p \sum_j [L_{K_j} L_{K_j}, Q]$$

is a differential operator of order  $m+1$ . Moreover, by (2.21), (2.34), the leading term of order  $m-1$  differential operator in  $[L_{K_j} L_{K_j}, Q]$  is  $p^2[(\Gamma^L - 2\pi\sqrt{-1}\mu)(K_j))^2, Q]$ . Thus by (2.39) and (2.40),

$$(2.41) \quad \begin{aligned} \|Qs\|_{H_p^2} &\leq C(\|\mathcal{L}_p Qs\|_{L^2} + p^2\|Qs\|_{L^2}) \\ &\leq C(\|Q\mathcal{L}_p s\|_{L^2} + p\|s\|_{H_p^{m+1}} + p^2\|s\|_{H_p^m} + p^3\|s\|_{H_p^{m-1}}). \end{aligned}$$



This means

$$(2.42) \quad \|s\|_{H_p^{2m+2}} \leq C_m p^{2m+2} \sum_{j=0}^{m+1} \|\mathcal{L}_p^j s\|_{L^2}.$$

Moreover, from

$$\langle \mathcal{L}_p^{m'} \phi_p(\mathcal{L}_p) Qs, s' \rangle = \langle s, Q^* \phi_p(\mathcal{L}_p) \mathcal{L}_p^{m'} s' \rangle,$$

(2.31) and (2.33), we know that for any  $l, m' \in \mathbb{N}$ , there exists  $C_{l,m'} > 0$  such that for  $p \geq 1$ ,

$$(2.43) \quad \|\mathcal{L}_p^{m'} \phi_p(\mathcal{L}_p) Qs\|_{L^2} \leq C_{l,m'} p^{-l+m} \|s\|_{L^2}.$$

We deduce from (2.42) and (2.43) that if  $Q_1, Q_2$  are differential operators of order  $m, m'$  with compact support in  $U_i, U_j$  respectively, then for any  $l > 0$ , there exists  $C_l > 0$  such that for  $p \geq 1$ ,

$$(2.44) \quad \|Q_1 \phi_p(\mathcal{L}_p) Q_2 s\|_{L^2} \leq C_l p^{-l} \|s\|_{L^2}.$$

On  $U_i \times U_j$ , by using Sobolev inequality and (2.32), we get Proposition 2.6.  $\square$

Observe that  $K_j^X$  are vector fields along the orbits of the  $G$ -action, thus the contribution of  $pL_{K_j}L_{K_j}$  in the wave operator  $\exp(\sqrt{-1}t\sqrt{\mathcal{L}_p})$  will propagate along the  $G$ -orbits, and the principal symbol of  $\mathcal{L}_p$  is given by

$$\sigma(\mathcal{L}_p)(\xi) = |\xi|^2 + p \sum_j \langle K_j^X, \xi \rangle^2 \quad \text{for } \xi \in T^*X.$$

By the finite propagation speed for solutions of hyperbolic equations [16, §7.8], [37, §4.4], [38, I. §2.6, §2.8],  $\tilde{F}(\mathcal{L}_p)(x, x')$  only depends on the restriction of  $\mathcal{L}_p$  to  $G \cdot B^X(x, \varepsilon)$  and

$$(2.45) \quad \tilde{F}(\mathcal{L}_p)(x, x') = 0, \quad \text{if } d^X(Gx, x') \geq \varepsilon.$$

(When we apply the proof of [38, §2.6, §2.8], we need to suppose that  $\Sigma_1, \Sigma_2$  therein are  $G$ -space-like surfaces for the operator  $\frac{\partial^2}{\partial t^2} - D_p^2$ ).

Combining with Proposition 2.6, we know that the asymptotic of  $P_p^G(x, x')$  as  $p \rightarrow \infty$  is localized on a neighborhood of  $Gx$ .

*Proof of Theorem 0.1.* From Proposition 2.6 and (2.45), we get (0.7) for any  $x, x' \in X$ ,  $d^X(Gx, x') \geq \varepsilon_0$ . Now we will establish (0.7) for  $x, x' \in X \setminus U$ .

Recall that  $U$  is a  $G$ -open neighborhood of  $P = \mu^{-1}(0)$ .

As 0 is a regular value of  $\mu$ , there exists  $\epsilon_0 > 0$  such that  $\mu : X_{2\epsilon_0} = \mu^{-1}(B^{\mathfrak{g}^*}(0, 2\epsilon_0)) \rightarrow B^{\mathfrak{g}^*}(0, 2\epsilon_0)$  is a submersion,  $X_{2\epsilon_0}$  is a  $G$ -open subset of  $X$ .

Fix  $\varepsilon, \epsilon_0 > 0$  small enough such that  $X_{2\epsilon_0} \subset U$ , and  $d^X(x, y) > 4\varepsilon$  for any  $x \in X_{\epsilon_0}$ ,  $y \in X \setminus U$ . Then  $V_{\epsilon_0} = X \setminus X_{\epsilon_0}$  is a smooth  $G$ -manifold with boundary  $\partial V_{\epsilon_0}$ .

Consider the operator  $\mathcal{L}_p$  on  $V_{\epsilon_0}$  with the Dirichlet boundary condition. We denote it by  $\mathcal{L}_{p,D}$ . Note that  $\mathcal{L}_{p,D}$  is self-adjoint.

Still from [38, §2.6, §2.8], the wave operator  $\exp(\sqrt{-1}t\sqrt{\mathcal{L}_{p,D}})$  is well defined and  $\exp(\sqrt{-1}t\sqrt{\mathcal{L}_{p,D}})(x, x')$  only depends on the restriction of  $\mathcal{L}_p$  to  $G \cdot B^X(x, t) \cap V_{\epsilon_0}$ , and is zero if  $d^X(Gx, x') \geq t$ . Thus, by (2.29),

$$(2.46) \quad \tilde{F}(\mathcal{L}_p)(x, x') = \tilde{F}(\mathcal{L}_{p,D})(x, x'), \quad \text{if } x, x' \in X \setminus U.$$

Now for  $s \in \mathcal{C}_0^\infty(V_{\epsilon_0}, E_p)$ , after taking an integration over  $G$ , we can get the decomposition  $s = s_1 + s_2$  with  $s_1 \in \Omega^{0,\bullet}(X, L^p \otimes E)^G$ ,  $s_2 \in (\Omega^{0,\bullet}(X, L^p \otimes E)^G)^\perp$  and  $\text{supp } s_i \subset V_{\epsilon_0} \setminus \partial V_{\epsilon_0}$ .

Since  $\sum_{i=1}^{\dim G} L_{K_i} L_{K_i}$  commutes with the  $G$ -action,  $\mathcal{L}_p s_1 \in \Omega^{0,\bullet}(X, L^p \otimes E)^G$ ,  $\mathcal{L}_p s_2 \in (\Omega^{0,\bullet}(X, L^p \otimes E)^G)^\perp$  and, by (2.22), (2.26),

$$(2.47) \quad \begin{aligned} \langle \mathcal{L}_p s, s \rangle &= \langle \mathcal{L}_p s_1, s_1 \rangle + \langle \mathcal{L}_p s_2, s_2 \rangle \\ &= \|D_p s_2\|_{L^2}^2 - p \langle \rho(\text{Cas}) s_2, s_2 \rangle + \langle D_p^2 s_1, s_1 \rangle \\ &\geq p\nu_1 \|s_2\|_{L^2}^2 + \langle D_p^2 s_1, s_1 \rangle. \end{aligned}$$

To estimate the term  $\langle D_p^2 s_1, s_1 \rangle$ , we will use the Lichnerowicz formula.

Recall that the Bochner-Laplacian  $\Delta^{E_p}$  on  $E_p$  is defined by (1.21).

Let  $r^X$  be the Riemannian scalar curvature of  $(TX, g^{TX})$ .

Let  $\{w_a\}$  be an orthonormal frame of  $(T^{(1,0)}X, g^{TX})$ . Set

$$(2.48) \quad \begin{aligned} \omega_d &= - \sum_{a,b} R^L(w_a, \bar{w}_b) \bar{w}^b \wedge i_{\bar{w}_a}, \\ \tau(x) &= \sum_a R^L(w_a, \bar{w}_a), \quad R_\tau^E = \sum_a R^E(w_a, \bar{w}_a), \\ \mathbf{c}(R) &= \sum_{a < b} \left( R^E + \frac{1}{2} \text{Tr}[R^{T^{(1,0)}X}] \right) (e_a, e_b) c(e_a) c(e_b). \end{aligned}$$

The Lichnerowicz formula [1, Theorem 3.52] (cf. [25, Theorem 2.2]) for  $D_p^2$  is

$$(2.49) \quad D_p^2 = \Delta^{E_p} - 2p\omega_d - p\tau + \frac{1}{4}r^X + \mathbf{c}(R).$$

Especially, as  $\text{supp } s_i \subset V_{\epsilon_0} \setminus \partial V_{\epsilon_0}$ , from (2.49), we get

$$(2.50) \quad \langle D_p^2 s_1, s_1 \rangle = \|\nabla^{E_p} s_1\|_{L^2}^2 - p \langle (2\omega_d + \tau) s_1, s_1 \rangle + \langle (\frac{1}{4}r^X + \mathbf{c}(R)) s_1, s_1 \rangle.$$

Since  $s_1 \in \Omega^{0,\bullet}(X, L^p \otimes E)^G$ , from (1.13), for any  $K \in \mathfrak{g}$ ,

$$(2.51) \quad \nabla_{K^X}^{E_p} s_1 = (L_K + \mu^{E_p}(K)) s_1 = \mu^{E_p}(K) s_1.$$

From (2.21) and (2.51), there exist  $C, C' > 0$  such that

$$(2.52) \quad \begin{aligned} \|\nabla^{E_p} s_1\|_{L^2}^2 &\geq C \sum_j \|\nabla_{K_j^X}^{E_p} s_1\|_{L^2}^2 = C \sum_j \|\mu^{E_p}(K_j) s_1\|_{L^2}^2 \\ &\geq Cp^2 \|\mu|s_1\|_{L^2}^2 - C' \|s_1\|_{L^2}^2 \geq C\epsilon_0^2 p^2 \|s_1\|_{L^2}^2 - C' \|s_1\|_{L^2}^2. \end{aligned}$$

From (2.47)-(2.52), for  $p$  large enough,

$$(2.53) \quad \langle \mathcal{L}_p s, s \rangle \geq p\nu_1 \|s_2\|_{L^2}^2 + Cp^2 \|s_1\|_{L^2}^2.$$

Thus there are  $C, C' > 0$  such that for  $p \geq 1$ ,

$$(2.54) \quad \text{Spec}(\mathcal{L}_{p,D}) \subset [Cp - C', \infty[.$$

At first as  $K_j^X \in T\partial V_{\epsilon_0}$  for any  $j$ , thus  $L_{K_j}$  preserves the Dirichlet boundary condition. We get for  $l \in \mathbb{N}$ ,

$$(2.55) \quad L_{K_j} \phi_p(\mathcal{L}_{p,D}) = \phi_p(\mathcal{L}_{p,D}) L_{K_j}, \quad (\mathcal{L}_{p,D})^l \phi_p(\mathcal{L}_{p,D}) = \phi_p(\mathcal{L}_{p,D}) (\mathcal{L}_{p,D})^l.$$

Thus from (2.22) and (2.55),

$$(2.56) \quad D_{p,D}^2 \leq \mathcal{L}_{p,D},$$

and for  $l \in \mathbb{N}$ ,  $(D_{p,D}^2)^l$  commutes with the operator  $\phi_p(\mathcal{L}_{p,D})$ .

Let  $\phi_p(\mathcal{L}_{p,D})(x, x')$  be the smooth kernel of  $\phi_p(\mathcal{L}_{p,D})$  with respect to  $dv_X(x')$ .

Then from the above argument we get that for any  $l, k \in \mathbb{N}$ ,  $(D_{p,x}^2)^l (D_{p,x'}^2)^k \phi_p(\mathcal{L}_{p,D})(x, x')$  verifies the Dirichlet boundary condition for  $x, x'$  respectively.

By (2.34) and the elliptic estimate for Laplacian with Dirichlet boundary condition [38, Theorem 5.1.3], there exists  $C > 0$  such that for  $s \in H^{2m+2}(X, E_p) \cap H_0^1(X, E_p)$ ,  $p \in \mathbb{N}$ , we have

$$(2.57) \quad \|s\|_{H^{2m+2}} \leq C(\|D_p^2 s\|_{H^{2m}} + p^2 \|s\|_{H^{2m+1}}).$$

Thus if  $Q_1, Q_2$  are differential operators of order  $2m, 2m'$  with compact support in  $U_i, U_j$  respectively, by (2.57) and (2.56), as in (2.42), we get for  $s \in \mathcal{C}_0^\infty(V_{\epsilon_0}, E_p)$ ,

$$(2.58) \quad \|Q_1 \phi_p(\mathcal{L}_{p,D}) Q_2 s\|_{L^2} \leq Cp^{2m+2m'} \sum_{j_1=0}^m \sum_{j_2=0}^{m'} \|(D_{p,D}^2)^{j_1} \phi_p(\mathcal{L}_{p,D}) (D_{p,D}^2)^{j_2} s\|_{L^2} \\ \leq Cp^{2m+2m'} \sum_{j_1=0}^m \sum_{j_2=0}^{m'} \|(\mathcal{L}_{p,D})^{j_1} \phi_p(\mathcal{L}_{p,D}) (\mathcal{L}_{p,D})^{j_2} s\|_{L^2}.$$

From (2.54), (2.58), as in (2.44), we get

$$(2.59) \quad \|Q_1 \phi_p(\mathcal{L}_{p,D}) Q_2 s\|_{L^2} \leq C_l p^{-l} \|s\|_{L^2}.$$

By using Sobolev inequality as in the proof of Proposition 2.6, from (2.30), (2.46) and (2.59), we get Theorem 0.1.  $\square$

**2.5. Induced operator on  $U/G$ .** Let  $U$  be a  $G$ -neighborhood of  $P = \mu^{-1}(0)$  in  $X$  such that  $G$  acts freely on  $\overline{U}$ , the closure of  $U$ . We will use the notation as in Introduction and Sections 1.1, 1.2 with  $X$  therein replaced by  $U$ , especially  $B = U/G$ .

Let  $\pi : U \rightarrow B$  be the natural projection with fiber  $Y$ . Let  $TY$  be the sub-bundle of  $TU$  generated by the  $G$ -action, let  $g^{TY}, g^{TP}$  be the metrics on  $TY, TP$  induced by  $g^{TX}$ .

Let  $T^H U, T^H P$  be the orthogonal complements of  $TY$  in  $TU, (TP, g^{TP})$ . Let  $g^{T^H U}$  be the metric on  $T^H U$  induced by  $g^{TX}$ , and it induces naturally a Riemannian metric  $g^{TB}$  on  $B$ .

Let  $dv_B$  be the Riemannian volume form on  $(B, g^{TB})$ .

Recall that in (1.20), we defined the isometry  $\Phi = h\pi_G : (\mathcal{C}^\infty(U, E_p)^G, \langle, \rangle) \rightarrow (\mathcal{C}^\infty(B, E_{p,B}), \langle, \rangle)$ .

By (1.14),  $\mu^{E_p}$  defines a  $G$ -invariant section  $\tilde{\mu}^{E_p}$  of  $TY \otimes \text{End}(E_p)$  on  $U$ .

Remark that  $\omega_d, \tau, \mathbf{c}(R)$  in (2.48) are  $G$ -invariant. We still denote by  $\omega_d, \tau, \mathbf{c}(R)$  the induced sections on  $B$ .

As a direct corollary of Theorem 1.3 and (2.49), we get the following result,

**Proposition 2.7.** *As an operator on  $\mathcal{C}^\infty(B, E_{p,B})$ ,*

$$(2.60) \quad \Phi \mathcal{L}_p \Phi^{-1} = \Phi D_p^2 \Phi^{-1} \\ = \Delta^{E_{p,B}} - \langle \tilde{\mu}^{E_p}, \tilde{\mu}^{E_p} \rangle_{g^{TY}} - \frac{1}{h} \Delta_B h - 2p\omega_d - p\tau + \frac{1}{4}r^X + \mathbf{c}(R).$$

From Theorem 0.1, Prop. 2.6 and (2.45), modulo  $\mathcal{O}(p^{-\infty})$ ,  $P_p^G(x, x')$  depends only the restriction of  $\mathcal{L}_p$  on  $U$ .

To get a complete picture on  $P_p^G(x, x')$ , we explain now that modulo  $\mathcal{O}(p^{-\infty})$ ,  $P_p^G(x, x')$  depends only on the restriction of  $\Phi \mathcal{L}_p \Phi^{-1}$  on any neighborhood of  $X_G$  in  $B$ .

As in the proof of Theorem 0.1, we will fix  $\epsilon_0 > 0$  small enough such that  $X_{2\epsilon_0} = \mu^{-1}(B^{\mathfrak{g}*}(0, 2\epsilon_0)) \subset U$ , and the constant  $\varepsilon > 0$  which will be fixed later, verifying that  $d^X(x, y) > 4\varepsilon$  for any  $x \in X_{\epsilon_0}, y \in X \setminus U$ . Set  $B_{\epsilon_0} = \pi(X_{\epsilon_0})$ .

First we will extend all objects from a neighborhood of  $P$  to the total space of the normal bundle  $N$  of  $P$  in  $X$ .

Let  $\pi_N : N \rightarrow P$  be the normal bundle of  $P$  in  $X$ . We identify  $N$  to the orthogonal complement of  $TP$  in  $(TX, g^{TX})$ . Then  $G$  acts on  $N$  and the action extends naturally on  $\pi_N^*(L|_P), \pi_N^*(E|_P)$ .

By (2.5), we have an orthogonal decomposition of  $TX$ ,

$$(2.61) \quad TX|_P = T^H P \oplus TY|_P \oplus N, \quad \text{and } TY|_P \simeq P \times \mathfrak{g}, \quad N = \mathbf{J}TY|_P \simeq P \times \mathfrak{g}.$$

Denote by  $P^{TY}, P^{TP}, P^{N_P}$  the orthogonal projections from  $TX$  on  $TY, TP$  and  $N|_P$  by this identification.

From (2.61), we have

$$(2.62) \quad TN \simeq \pi_N^* TX|_P \simeq \pi_N^*(TP \oplus \mathfrak{g}).$$

For  $\varepsilon > 0$ , we denote by  $B_\varepsilon^N = \{(y, Z) \in N, y \in P, |Z|_{g^{TX}} \leq \varepsilon\}$ .

Then for  $\varepsilon_0$  small enough, the map  $(y, Z) \in N \rightarrow \exp_y^X(Z) \in X$  is a diffeomorphism from  $B_{2\varepsilon_0}^N$  onto a tubular neighborhood  $\mathcal{U}_{2\varepsilon_0}$  of  $P$  in  $X$ .

From now on, we use the notation  $(y, Z)$  instead of  $\exp_y^X(Z)$ . We identify  $y \in P$  with  $(y, 0) \in N$ . From (2.61), (2.62), we may and we will identify  $TN$  to  $\pi_N^* TP \oplus \mathfrak{g}$ .

For  $Z \in N_y, |Z| \leq 2\varepsilon_0$ , we identify  $L_Z, E_Z$  to  $L_y, E_y$  by using parallel transport with respect to  $\nabla^L, \nabla^E$  along the curve  $[0, 1] \ni u \rightarrow uZ$ . In this way, we identify the Hermitian bundles  $(\pi_N^* L|_P, \pi_N^* h^L), (\pi_N^* E|_P, \pi_N^* h^E)$  to  $(L, h^L), (E, h^E)$  on  $B_{2\varepsilon_0}^N$ .

Let  $\varepsilon > 0$  with  $\varepsilon < \varepsilon_0/2$ . Let  $\varphi : \mathbb{R} \rightarrow [0, 1]$  be a smooth even function such that

$$(2.63) \quad \varphi(v) = 1 \quad \text{if } |v| < 2; \quad \varphi(v) = 0 \quad \text{if } |v| > 4.$$

Let  $\psi_\varepsilon : N \rightarrow N$  be the map defined by  $\psi_\varepsilon(Z) = \varphi(|Z|/\varepsilon)Z \in N_y$  for  $Z \in N_y$ .

Let  $g_Z^{TX} = g_{\psi_\varepsilon(Z)}^{TX}, J_Z^N = J_{\psi_\varepsilon(Z)}^N$  be the induced metric and almost-complex structure on  $N$ .

Let  $\nabla^{\pi_N^*E} = \psi_\varepsilon^* \nabla^E$ , then  $\nabla^{\pi_N^*E}$  is the extension of  $\nabla^E$  on  $B_{2\varepsilon}^N$ .

Let  $\nabla^{\pi_N^*L}$  be the Hermitian connection on  $(\pi_N^*L, \pi_N^*h^L)$  defined by that for  $Z \in N_y$ ,

$$(2.64) \quad \nabla^{\pi_N^*L} = \psi_\varepsilon^* \nabla^L + (1 - \varphi(\frac{|Z|}{\varepsilon}))R_y^L(Z, P^{TP}\cdot) + \frac{1}{2}(1 - \varphi^2(\frac{|Z|}{\varepsilon}))R_y^L(Z, P^{N_P}\cdot).$$

Then by using the identification (2.61), and (2.62), we calculate directly that its curvature  $R^{\pi_N^*L} = (\nabla^{\pi_N^*L})^2$  is

$$(2.65) \quad \begin{aligned} R_Z^{\pi_N^*L} &= \psi_\varepsilon^* R^L + d\left((1 - \varphi(\frac{|Z|}{\varepsilon}))R_y^L(Z, P^{TP}\cdot) + \frac{1}{2}(1 - \varphi^2(\frac{|Z|}{\varepsilon}))R_y^L(Z, P^{N_P}\cdot)\right) \\ &= R_{\psi_\varepsilon(Z)}^L(P^{TP}\cdot, P^{TP}\cdot) + R_y^L(P^{N_P}\cdot, \cdot) + \varphi^2(\frac{|Z|}{\varepsilon})(R_{\psi_\varepsilon(Z)}^L - R_y^L)(P^{N_P}\cdot, P^{N_P}\cdot) \\ &\quad + \varphi(\frac{|Z|}{\varepsilon})(R_{\psi_\varepsilon(Z)}^L - R_y^L)(P^{N_P}\cdot, P^{TP}\cdot) \\ &\quad - \varphi'(\frac{|Z|}{\varepsilon})\frac{Z^*}{\varepsilon|Z|} \wedge [R_y^L(Z, P^{TP}\cdot) - R_{\psi_\varepsilon(Z)}^L(Z, P^{TP}\cdot)] \\ &\quad - (\varphi\varphi')(\frac{|Z|}{\varepsilon})\frac{Z^*}{\varepsilon|Z|} \wedge [R_y^L(Z, P^{N_P}\cdot) - R_{\psi_\varepsilon(Z)}^L(Z, P^{N_P}\cdot)] \\ &\quad + d_y\left((1 - \varphi(\frac{|Z|}{\varepsilon}))R_y^L(Z, P^{TP}\cdot) + \frac{1}{2}(1 - \varphi^2(\frac{|Z|}{\varepsilon}))R_y^L(Z, P^{N_P}\cdot)\right). \end{aligned}$$

Here  $Z^* \in N^*$  is the dual of  $Z \in N$  with respect to the metric  $g^N$ .

From (2.65), one deduces that  $R^{\pi_N^*L}$  is positive in the sense of (2.9) when  $\varepsilon$  is small enough, with the corresponding constant  $\nu_0$  for  $R^{\pi_N^*L}$  being larger than  $\frac{4}{5}\nu_0$ .

Note that  $G$  acts naturally on the normal bundle  $N$ , and under our identification, the  $G$ -actions on  $L, E$  on  $B_\varepsilon^N$  are exactly the  $G$ -actions on  $L|_P, E|_P$  on  $P$ .

Now we define the  $G$ -actions on  $\pi_N^*L, \pi_N^*E$  by their  $G$ -actions on  $P$ , then they extend the  $G$ -actions on  $L, E$  on  $B_\varepsilon^N$  to  $N$ .

By (2.14), the moment map  $\mu_N : N \rightarrow \mathfrak{g}^*$  of the  $G$ -action on  $N$  is defined by

$$(2.66) \quad -2\pi\sqrt{-1}\mu_N(K) = L_K - \nabla_{K^N}^{\pi_N^*L}, \quad K \in \mathfrak{g}.$$

Observe that  $\psi_{\varepsilon*}K_{(y,Z)}^N = K_y^P \in TP$ , thus from (2.14), (2.61), and (2.64),

$$(2.67) \quad \begin{aligned} 2\pi\sqrt{-1}\mu_N(K)_{(y,Z)} &= (1 - \varphi(|Z|/\varepsilon))R_y^L(Z, K^P) + 2\pi\sqrt{-1}\mu(K)_{\psi_\varepsilon(Z)} \\ &= R_y^L(Z, K^P) + \mathcal{O}(\varphi^2(|Z|/\varepsilon)|Z|^2). \end{aligned}$$

Thus  $\mu_N^{-1}(0) = \mu^{-1}(0) = P$  for  $\varepsilon$  small enough, and for  $|Z| \geq 4\varepsilon$ ,

$$(2.68) \quad 2\pi\sqrt{-1}\mu_N(K)_{(y,Z)} = R_y^L(Z, K^P).$$

From now on, we fix  $\varepsilon$  as above.

Let  $\tilde{F}(\Phi\mathcal{L}_p\Phi^{-1})(x, x')$  ( $x, x' \in B_{\varepsilon_0}$ ) be the smooth kernel of  $\tilde{F}(\Phi\mathcal{L}_p\Phi^{-1})$  with respect to  $dv_B(x')$ . We will also view  $\tilde{F}(\Phi\mathcal{L}_p\Phi^{-1})$  as a  $G \times G$ -invariant section of  $\text{pr}_1^*E_p \otimes \text{pr}_2^*E_p^*$  on  $X_{\varepsilon_0} \times X_{\varepsilon_0}$ .

**Theorem 2.8.** *For any  $l, m \in \mathbb{N}$ , there exists  $C_{l,m} > 0$  such that for  $p \geq 1$ ,  $x, x' \in X_{\varepsilon_0}$ ,*

$$(2.69) \quad |h(x)h(x')P_p^G(x, x') - \tilde{F}(\Phi\mathcal{L}_p\Phi^{-1})(\pi(x), \pi(x'))|_{\mathcal{C}^m(X_{\varepsilon_0} \times X_{\varepsilon_0})} \leq C_{l,m}p^{-l}.$$

*Proof.* Let  $D_p^N$  be the Dirac operator on  $N$  associated to the above data by the construction in Section 2.2. By the argument in [25, p. 656-657] and the proof of Theorem 2.5, we know that Theorems 2.2, 2.5 still hold for  $D_p^N$ .

Let  $\mathcal{L}_p^N$  be the operator on  $N$  defined as in (2.22). Then there exists  $C > 0$  such that for  $p \geq 1$ ,

$$(2.70) \quad \text{Spec}(\mathcal{L}_p^N) \subset \{0\} \cup [p\nu - C, +\infty[.$$

Let  $P_p^{N,G}$  be the orthogonal projection from  $\Omega^{0,\bullet}(N, \pi_N^*(L^p \otimes E))$  on  $(\text{Ker } D_p^N)^G$ , then by (2.70) and the arguments as in the proof of Theorem 2.6, for any  $l, m \in \mathbb{N}$ ,  $V \subset N$  a compact subset of  $N$ , there exists  $C_{l,m} > 0$  such that for  $p \geq 1$ ,  $x, x' \in V$ ,

$$(2.71) \quad |\tilde{F}(\mathcal{L}_p^N)(x, x') - P_p^{N,G}(x, x')|_{\mathcal{C}^m(V \times V)} \leq C_{l,m} p^{-l}.$$

Let  $P_p^{N/G}$  be the projection from  $(L^2(N/G, (\Lambda(T^{*(0,1)}N) \otimes \pi_N^*(L^p \otimes E))_{N/G}), \langle \cdot, \cdot \rangle)$  onto  $\text{Ker}(\Phi \mathcal{L}_p^N \Phi^{-1})$ , and let  $P_p^{N/G}(z, z')$  be the smooth kernel of the operator  $P_p^{N/G}$  with respect to  $dv_{N/G}(z')$ .

We still denote by  $\text{pr}_1, \text{pr}_2$  the projections from  $N \times N$  onto the first and second factor  $N$ . We will also view  $P_p^{N/G}(z, z')$  as a  $G \times G$ -invariant section of

$$\text{pr}_1^*(\Lambda(T^{*(0,1)}N) \otimes \pi_N^*(L^p \otimes E)) \otimes \text{pr}_2^*(\Lambda(T^{*(0,1)}N) \otimes \pi_N^*(L^p \otimes E))^*$$

on  $N \times N$ .

As  $\Phi$  in (1.20) defines an isometry from  $(\text{Ker } D_p^N)^G = \text{Ker } \mathcal{L}_p^N$  onto  $\text{Ker}(\Phi \mathcal{L}_p^N \Phi^{-1})$ , one has

$$(2.72) \quad h(x)h(x')P_p^{N,G}(x, x') = P_p^{N/G}(\pi(x), \pi(x')).$$

On  $N/G$ , by the arguments as in the proof of Theorem 2.6, we get

$$(2.73) \quad |\tilde{F}(\Phi \mathcal{L}_p \Phi^{-1})(z, z') - P_p^{N/G}(z, z')|_{\mathcal{C}^m(V/G \times V/G)} \leq C_{l,m} p^{-l}.$$

By the finite propagation speed (2.45), we know that for  $x, x' \in X_{\epsilon_0}$ ,

$$(2.74) \quad \tilde{F}(\mathcal{L}_p^N)(x, x') = \tilde{F}(\mathcal{L}_p)(x, x').$$

Now we get (2.69) from (2.30), (2.71)-(2.74). □

Let  $d^B(\cdot, \cdot)$  be the Riemannian distance on  $B$ .

By (2.60) and the finite propagation speed for solutions of hyperbolic equations [16, §7.8], [37, §4.4],  $\tilde{F}(\Phi \mathcal{L}_p \Phi^{-1})(x, x')$  only depends on the restriction of  $\Phi \mathcal{L}_p \Phi^{-1}$  to  $B^B(x, \varepsilon)$  and

$$(2.75) \quad \tilde{F}(\Phi \mathcal{L}_p \Phi^{-1})(x, x') = 0, \quad \text{if } d^B(x, x') \geq \varepsilon.$$

Thus we have localized our problem near  $X_G$ .

Theorem 2.8 helps us to understand that the asymptotic of  $P_p^G(x, x')$  is local near  $X_G$ . In the rest, we will not use directly Theorem 2.8, but the argument of its proof will be used in Section 2.6.

**2.6. Rescaling and a Taylor expansion of the operator  $\Phi\mathcal{L}_p\Phi^{-1}$ .** Recall that  $N_G$  is the normal bundle of  $X_G$  in  $B$ , and we identify  $N_G$  as the orthogonal complement of  $TX_G$  in  $(TB, g^{TB})$ .

Let  $P^{TX_G}, P^{N_G}$  be the orthogonal projection from  $TB$  on  $TX_G$ ,  $N_G$  on  $X_G$ .

Recall that  $\nabla^{N_G}, {}^0\nabla^{TB}$  are connections on  $N_G$ ,  $TB$  on  $X_G$ , and  $A$  is the associated second fundamental form defined in (0.9).

We fix  $x_0 \in X_G$ .

If  $W \in T_{x_0}X_G$ , let  $\mathbb{R} \ni t \rightarrow x_t = \exp_{x_0}^{X_G}(tW) \in X_G$  be the geodesic in  $X_G$  such that  $x_t|_{t=0} = x_0$ ,  $\frac{dx}{dt}|_{t=0} = W$ .

If  $W \in T_{x_0}X_G$ ,  $|W| \leq \varepsilon$ ,  $V \in N_{x_0}$ , let  $\tau_W V \in N_{G, \exp_{x_0}^{X_G}(W)}$  be the natural parallel transport of  $V$  with respect to the connection  $\nabla^{N_G}$  along the curve  $[0, 1] \ni t \rightarrow \exp_{x_0}^{X_G}(tW)$ .

If  $Z \in T_{x_0}B$ ,  $Z = Z^0 + Z^\perp$ ,  $Z^0 \in T_{x_0}X_G$ ,  $Z^\perp \in N_{x_0}$ ,  $|Z^0|, |Z^\perp| \leq \varepsilon$ , we identify  $Z$  with  $\exp_{\exp_{x_0}^{X_G}(Z^0)}^{B_{x_0}(Z^0)}(\tau_{Z^0}Z^\perp)$ . This identification is a diffeomorphism from  $B_{x_0}^{TX_G}(0, \varepsilon) \times B_{x_0}^{N_G}(0, \varepsilon)$  into an open neighborhood  $\mathcal{U}(x_0)$  of  $x_0$  in  $B$ . We denote it by  $\Psi$ , and  $\mathcal{U}(x_0) \cap X_G = B_{x_0}^{TX_G}(0, \varepsilon) \times \{0\}$ .

From now on, we use indifferently the notation  $B_{x_0}^{TX_G}(0, \varepsilon) \times B_{x_0}^{N_G}(0, \varepsilon)$  or  $\mathcal{U}(x_0)$ ,  $x_0$  or  $0, \dots$ .

We identify  $(L_B)_Z, (E_B)_Z$  and  $(E_{p,B})_Z$  to  $(L_B)_{x_0}, (E_B)_{x_0}$  and  $(E_{p,B})_{x_0}$  by using parallel transport with respect to  $\nabla^{L_B}, \nabla^{E_B}$  and  $\nabla^{E_{p,B}}$  along the curve  $\gamma_u : [0, 1] \ni u \rightarrow uZ$ .

Recall that  $T^H U \subset TX$  is the horizontal bundle for  $\pi : U \rightarrow B$  defined in Section 2.5.

Let  $P^{T^H U}$  be the orthogonal projection from  $TX$  onto  $T^H U$ .

For  $W \in TB$ , let  $W^H \in T^H U$  be the lift of  $W$ .

For  $y_0 \in \pi^{-1}(x_0)$ , we define the curve  $\tilde{\gamma}_u : [0, 1] \rightarrow X$  to be the lift of the curve  $\gamma_u$  with  $\tilde{\gamma}_0 = y_0$  and  $\frac{\partial \tilde{\gamma}_u}{\partial u} \in T^H U$ . Then on  $\pi^{-1}(B^{TB}(0, \varepsilon))$ , we use the parallel transport with respect to  $\nabla^L, \nabla^E$  and  $\nabla^{E_p}$  along the curve  $\tilde{\gamma}_u$  to trivialized the corresponding bundles. By (1.17), the previous trivialization is naturally induced by this one.

Let  $\{e_i^0\}, \{e_j^\perp\}$  be orthonormal basis of  $T_{x_0}X_G$ ,  $N_{G, x_0}$ , then  $\{e_i\} = \{e_i^0, e_j^\perp\}$  is an orthonormal basis of  $T_{x_0}B$ . Let  $\{e^i\}$  be its dual basis. We will also denote  $\Psi_*(e_i^0), \Psi_*(e_j^\perp)$  by  $e_i^0, e_j^\perp$ . Thus in our coordinate,

$$(2.76) \quad \frac{\partial}{\partial Z_i^0} = e_i^0, \quad \frac{\partial}{\partial Z_j^\perp} = e_j^\perp.$$

For  $\varepsilon > 0$  small enough, we will extend the geometric objects on  $B^{TB}(x_0, \varepsilon)$  to  $\mathbb{R}^{2n-n_0} \simeq T_{x_0}B$  (here we identify  $(Z_1, \dots, Z_{2n-n_0}) \in \mathbb{R}^{2n-n_0}$  to  $\sum_i Z_i e_i \in T_{x_0}B$ ) such that  $D_p$  will become the restriction of a  $\text{spin}^c$  Dirac operator on  $G \times \mathbb{R}^{2n-n_0}$  associated to a Hermitian line bundle with positive curvature. In this way, we can replace  $X$  by  $G \times \mathbb{R}^{2n-n_0}$ .

First of all, we denote by  $L_0, E_0$  the trivial bundles  $L|_{G_{y_0}}, E|_{G_{y_0}}$  on  $X_0 = G \times \mathbb{R}^{2n-n_0}$ , and we still denote by  $\nabla^L, \nabla^E, h^L$  etc. the connections and metrics on  $L_0, E_0$  on  $\pi^{-1}(B^{T_{x_0}B}(0, 4\varepsilon))$  induced by the above identification. Then  $h^L, h^E$  is identified with the constant metrics  $h^{L_0} = h^{L_{y_0}}, h^{E_0} = h^{E_{y_0}}$ .

Set

$$(2.77) \quad \mathcal{R}^\perp = \sum_j Z_j^\perp e_j^\perp = Z^\perp, \quad \mathcal{R}^0 = \sum_i Z_i^0 e_i^0 = Z^0, \quad \mathcal{R} = \mathcal{R}^\perp + \mathcal{R}^0 = Z.$$

Then  $\mathcal{R}$  is the radial vector field on  $\mathbb{R}^{2n-n_0}$ .

Let  $\varphi_\varepsilon : X_0 \rightarrow X_0$  be the map defined by  $\varphi_\varepsilon(g, Z) = (g, \varphi(|Z|/\varepsilon)Z)$  for  $(g, Z) \in G \times \mathbb{R}^{2n-n_0}$ .

Let  $g^{TX_0}(g, Z) = g^{TX}(\varphi_\varepsilon(g, Z))$ ,  $J_0(g, Z) = J(\varphi_\varepsilon(g, Z))$  be the metric and almost-complex structure on  $X_0$ .

Let  $\nabla^{E_0} = \varphi_\varepsilon^* \nabla^E$ , then  $\nabla^{E_0}$  is the extension of  $\nabla^E$  on  $\pi^{-1}(B^{T_{x_0}B}(0, \varepsilon))$ .

Let  $\nabla^{L_0}$  be the Hermitian connection on  $(L_0, h^{L_0})$  on  $G \times \mathbb{R}^{2n-n_0}$  defined by for  $Z \in \mathbb{R}^{2n-n_0}$ ,

$$(2.78) \quad \nabla^{L_0} = \varphi_\varepsilon^* \nabla^L + \left(1 - \varphi\left(\frac{|Z|}{\varepsilon}\right)\right) R_{y_0}^L(\mathcal{R}^H, P_{y_0}^{TY} \cdot) + \frac{1}{2} \left(1 - \varphi^2\left(\frac{|Z|}{\varepsilon}\right)\right) R_{y_0}^L(\mathcal{R}^H, P_{y_0}^{T^H U} \cdot).$$

As in (2.65), its curvature  $R^{L_0}$  is positive in the sense of (2.9) for  $\varepsilon$  small enough, and the corresponding constant  $\nu_0$  for  $R^{L_0}$  is bigger than  $\frac{4}{5}\nu_0$  uniformly for  $y_0 \in P$ .

From now on, we fix  $\varepsilon$  as above.

Now  $G$  acts naturally on  $X_0$ , and under our identification, the  $G$ -action on  $L, E$  on  $G \times B^{T_{x_0}B}(0, \varepsilon)$  is exactly the  $G$ -action on  $L|_{Gy_0}, E|_{Gy_0}$ .

We define a  $G$ -action on  $L_0, E_0$  by its  $G$ -action on  $Gy_0$ , then it extends the  $G$ -action on  $L, E$  on  $G \times B^{T_{x_0}B}(0, \varepsilon)$  to  $X_0$ .

By (2.15), for any  $K \in \mathfrak{g}$ ,  $W \in TP$  on  $P = \mu^{-1}(0)$ , we have

$$(2.79) \quad \begin{aligned} R^L(W, K^X) &= -2\pi\sqrt{-1}\omega(W, K^X) = 2\pi\sqrt{-1}W(\mu(K)) = 0, \\ R_{(1, Z^0)}^L(\mathcal{R}^H, K^X) &= R_{(1, Z^0)}^L((\mathcal{R}^\perp)^H, K^X). \end{aligned}$$

Observe that for  $(1, Z) \in G \times \mathbb{R}^{2n-n_0}$ ,  $\varphi_{\varepsilon*} K_{(1, Z)}^{X_0} = K_{y_0}^X$  for  $K \in \mathfrak{g}$ , by (2.14), the moment map  $\mu_{X_0} : X_0 \rightarrow \mathfrak{g}^*$  of the  $G$ -action on  $X_0$  is given by

$$(2.80) \quad 2\pi\sqrt{-1}\mu_{X_0}(K)_{(1, Z)} = (1 - \varphi(\frac{|Z|}{\varepsilon})) R_{y_0}^L(\mathcal{R}^H, K_{y_0}^X) + 2\pi\sqrt{-1}\mu(K)_{\varphi_\varepsilon(1, Z)}.$$

Now from the choice of our coordinate, we know that  $\mu_{X_0} = 0$  on  $G \times \mathbb{R}^{2n-2n_0} \times \{0\}$ . Moreover,

$$(2.81) \quad 2\pi\sqrt{-1}\mu(K)_{\varphi_\varepsilon(1, Z)} = R_{(1, Z)}^L(\varphi(\frac{|Z|}{\varepsilon})(\mathcal{R}^\perp)^H, K^X) + \mathcal{O}(\varphi(\frac{|Z|}{\varepsilon})|Z||Z^\perp|).$$

From our construction, (2.80) and (2.81), we know that

$$(2.82) \quad \mu_{X_0}^{-1}(0) = G \times \mathbb{R}^{2n-2n_0} \times \{0\}.$$

By (2.79) and (2.80), for  $Z \in T_{x_0}B$ ,  $|Z| \geq 4\varepsilon$ ,

$$(2.83) \quad 2\pi\sqrt{-1}\mu_{X_0}(K)_{(1, Z)} = R_{y_0}^L((\mathcal{R}^\perp)^H, K_{y_0}^X).$$

Let  $D_p^{X_0}$  be the Dirac operator on  $X_0$  associated to the above data by the construction in Section 2.2. As in (2.70), the analogue of Theorems 2.2, 2.5 still holds for  $D_p^{X_0}$ .

Let  $g^{TB_0}$  be the metric on  $B_0 = \mathbb{R}^{2n-n_0}$  induced by  $g^{TX_0}$ , and let  $dv_{B_0}$  be the Riemannian volume form on  $(B_0, g^{TB_0})$ .

The operator  $\Phi \mathcal{L}_p^{X_0} \Phi^{-1}$  is also well-defined on  $T_{x_0}B \simeq \mathbb{R}^{2n-n_0}$ .



Let  $P_{x_0,p}$  be the orthogonal projection from  $L^2(\mathbb{R}^{2n-n_0}, (\Lambda(T^{*(0,1)}X_0) \otimes L_0^p \otimes E_0)_{B_0})$  onto  $\text{Ker}(\Phi \mathcal{L}_p^{X_0} \Phi^{-1})$  on  $\mathbb{R}^{2n-n_0}$ . Let  $P_{x_0,p}(Z, Z')$  ( $Z, Z' \in \mathbb{R}^{2n-n_0}$ ) be the smooth kernel of  $P_{x_0,p}$  with respect to  $dv_{B_0}(Z')$ . As before, we view  $P_{x_0,p}$  as a  $G \times G$ -invariant section of

$$\text{pr}_1^*(\Lambda(T^{*(0,1)}X_0) \otimes L_0^p \otimes E_0) \otimes \text{pr}_2^*(\Lambda(T^{*(0,1)}X_0) \otimes L_0^p \otimes E_0)^*$$

on  $X_0 \times X_0$ .

Let  $P_{0,p}^G$  be the orthogonal projection from  $\Omega^{0,\bullet}(X_0, L_0^p \otimes E_0)$  onto  $(\text{Ker } D_p^{X_0})^G$ , and let  $P_{0,p}^G(x, x')$  be the smooth kernel of  $P_{0,p}^G$  with respect to the volume form  $dv_{X_0}(x')$ .

Note that  $\Phi$  in (1.20) defines an isometry from  $(\text{Ker } D_p^{X_0})^G = \text{Ker } \mathcal{L}_p^{X_0}$  onto  $\text{Ker}(\Phi \mathcal{L}_p^{X_0} \Phi^{-1})$ , as in (2.72), we get

$$(2.84) \quad h(x)h(x')P_{0,p}^G(x, x') = P_{x_0,p}(\pi(x), \pi(x')).$$

**Proposition 2.9.** *For any  $l, m \in \mathbb{N}$ , there exists  $C_{l,m} > 0$  such that for  $x, x' \in G \times B^{T_{x_0}B}(0, \varepsilon)$ ,*

$$(2.85) \quad \left| (P_{0,p}^G - P_p^G)(x, x') \right|_{\mathcal{C}^m} \leq C_{l,m} p^{-l}.$$

*Proof.* By the analogue of Theorems 2.2, 2.5, we know that for  $x, x' \in G \times B^{T_{x_0}B}(0, \varepsilon)$ ,  $P_{0,p}^G - \tilde{F}(\mathcal{L}_p^{X_0})$  verifies also (2.30), and for  $x, x' \in G \times B^{T_{x_0}B}(0, \varepsilon)$ ,

$$\tilde{F}(\mathcal{L}_p^{X_0})(x, x') = \tilde{F}(\mathcal{L}_p)(x, x')$$

by finite propagation speed. Thus we get (2.85).  $\square$

Let  $T^{*(0,1)}X_0$  be the anti-holomorphic cotangent bundle of  $(X_0, J_0)$ . Since  $J_0(g, Z) = J(\varphi_\varepsilon(g, Z))$ ,  $T_{Z,J_0}^{*(0,1)}X_0$  is naturally identified with  $T_{\varphi_\varepsilon(g,Z),J}^{*(0,1)}X_0$ .

Let  $\nabla^{\text{Cliff}_0}$  be the Clifford connection on  $\Lambda(T^{*(0,1)}X_0)$  induced by the Levi-Civita connection  $\nabla^{TX_0}$  on  $(X_0, g^{TX_0})$ . Let  $R^{E_0}, R^{TX_0}, R^{\text{Cliff}_0}$  be the corresponding curvatures on  $E_0, TX_0$  and  $\Lambda(T^{*(0,1)}X_0)$  (cf. (2.10)).

We identify  $\Lambda(T^{*(0,1)}X_0)_{(g,Z)}$  with  $\Lambda(T_{(g,0)}^{*(0,1)}X)$  by identifying first  $\Lambda(T^{*(0,1)}X_0)_{(g,Z)}$  with  $\Lambda(T_{\varphi_\varepsilon(g,Z),J}^{*(0,1)}X_0)$ , which in turn is identified with  $\Lambda(T_{Gy_0}^{*(0,1)}X)$  by using parallel transport along  $u \rightarrow u\varphi_\varepsilon(g, Z)$  with respect to  $\nabla^{\text{Cliff}_0}$ . We also trivialize  $\Lambda(T^{*(0,1)}X_0)$  in this way.

Let  $S_L$  be a  $G$ -invariant unit section of  $L|_{Gy_0}$ . Using  $S_L$  and the above discussion, we get an isometry

$$\Lambda(T^{*(0,1)}X_0) \otimes E_0 \otimes L_0^p \simeq (\Lambda(T^{*(0,1)}X) \otimes E)|_{\pi^{-1}(x_0)} = \mathbf{E}|_{\pi^{-1}(x_0)}.$$

For any  $1 \leq i \leq 2n - n_0$ , let  $\tilde{e}_i(Z)$  be the parallel transport of  $e_i$  with respect to the connection  ${}^0\nabla^{TB}$  along  $[0, 1] \ni u \rightarrow uZ^0$ , and with respect to the connection  $\nabla^{TB}$  along  $[1, 2] \ni u \rightarrow Z^0 + (u - 1)Z^\perp$ .

If  $\alpha = (\alpha_1, \dots, \alpha_{2n-n_0})$  is a multi-index, set  $Z^\alpha = Z_1^{\alpha_1} \dots Z_{2n-n_0}^{\alpha_{2n-n_0}}$ .

Recall that  $A, \mathcal{R}^\perp$  have been defined in (0.9), (2.77).

The following Lemma extends [1, Prop. 1.28] (cf. also [17, Lemma 4.5]).

**Lemma 2.10.** *The Taylor expansion of  $\tilde{e}_i(Z)$  with respect to the basis  $\{e_i\}$  to order  $r$  is a polynomial of the Taylor expansion of the curvature coefficients of  $R^{TB}$  to order  $r - 2$  and  $A$  to order  $r - 1$ .*

*Proof.* Let  $\partial_i = \nabla_{e_i}$  be the partial derivatives along  $e_i$ .

Let  $\Gamma^{TB}$  be the connection form of  $\nabla^{TB}$  with respect to the frame  $\{\tilde{e}_i\}$  of  $TB$ . By the definition of our fixed frame, we have  $i_{\mathcal{R}^\perp} \Gamma^{TB} = 0$ . As in [1, (1.12)],

$$(2.86) \quad L_{\mathcal{R}^\perp} \Gamma^{TB} = [i_{\mathcal{R}^\perp}, d] \Gamma^{TB} = i_{\mathcal{R}^\perp} (d\Gamma^{TB} + \Gamma^{TB} \wedge \Gamma^{TB}) = i_{\mathcal{R}^\perp} R^{TB}.$$

Let  $\Theta(Z) = (\theta_j^i(Z))_{i,j=1}^{2n-n_0}$  be the  $(2n - n_0) \times (2n - n_0)$ -matrix such that

$$(2.87) \quad e_i = \sum_j \theta_j^i(Z) \tilde{e}_j(Z), \quad \tilde{e}_j(Z) = (\Theta(Z)^{-1})_j^k e_k.$$

Set  $\theta^j(Z) = \sum_i \theta_j^i(Z) e^i$  and

$$(2.88) \quad \theta = \sum_j e^j \otimes e_j = \sum_j \theta^j \tilde{e}_j \in T^*B \otimes TB.$$

As  $\nabla^{TB}$  is torsion free,  $\nabla^{TB} \theta = 0$ . Thus the  $\mathbb{R}^{2n-n_0}$ -valued one-form  $\theta = (\theta^j(Z))$  satisfies the structure equation,

$$(2.89) \quad d\theta + \Gamma^{TB} \wedge \theta = 0.$$

By the same proof of [1, Prop. 1.27], we have

$$(2.90) \quad \mathcal{R}^\perp = \sum_j Z_j^\perp \tilde{e}_j^\perp(Z), \quad i_{\mathcal{R}^\perp} \theta = \sum_j Z_j^\perp e_j^\perp = Z^\perp.$$

Here under our trivialization by  $\{\tilde{e}_i\}$ , we consider  $Z^\perp = (0, Z_1^\perp, \dots, Z_{n_0}^\perp)$  as a  $\mathbb{R}^{2n-n_0}$ -valued function.

Substituting (2.90) and  $(L_{\mathcal{R}^\perp} - 1)Z^\perp = 0$  into the identity  $i_{\mathcal{R}^\perp}(d\theta + \Gamma^{TB} \wedge \theta) = 0$ , we obtain

$$(2.91) \quad (L_{\mathcal{R}^\perp} - 1)L_{\mathcal{R}^\perp} \theta = (L_{\mathcal{R}^\perp} - 1)(dZ^\perp + \Gamma^{TB} Z^\perp) = (L_{\mathcal{R}^\perp} \Gamma^{TB}) Z^\perp = (i_{\mathcal{R}^\perp} R^{TB}) Z^\perp.$$

Here we consider  $R^{TB}$  as a matrix of 2-forms, so that  $R^{TB} Z^\perp$  is a vector of 2-forms, and  $\theta$  is a  $\mathbb{R}^{2n-n_0}$ -valued 1-form.

By (2.90) and (2.91), we get

$$(2.92) \quad i_{e_j} (L_{\mathcal{R}^\perp} - 1) L_{\mathcal{R}^\perp} \theta^i(Z) = \langle R^{TB}(\mathcal{R}^\perp, e_j) \mathcal{R}^\perp, \tilde{e}_i \rangle(Z).$$

We will denote by  $\partial^\perp, \partial^0$  the partial derivatives along  $N_G, TX_G$  respectively. Then we have the following Taylor expansions of (2.92): for  $j \in \{2(n - n_0) + 1, \dots, 2n - n_0\}$ , i.e.  $e_j \in N_G$ , by  $L_{\mathcal{R}^\perp} e^j = e^j$ , we have

$$(2.93) \quad \sum_{|\alpha^\perp| \geq 1} (|\alpha^\perp|^2 + |\alpha^\perp|) ((\partial^\perp)^{\alpha^\perp} \theta_j^i)(Z^0) \frac{(Z^\perp)^{\alpha^\perp}}{\alpha^\perp!} = \langle R^{TB}(\mathcal{R}^\perp, e_j) \mathcal{R}^\perp, \tilde{e}_i \rangle(Z).$$

and for  $j \in \{1, \dots, 2(n - n_0)\}$ , i.e.  $e_j \in TX_G$ , by  $L_{\mathcal{R}^\perp} e^j = 0$ , we have

$$(2.94) \quad \sum_{|\alpha^\perp| \geq 1} (|\alpha^\perp|^2 - |\alpha^\perp|) ((\partial^\perp)^{\alpha^\perp} \theta_j^i)(Z^0) \frac{(Z^\perp)^{\alpha^\perp}}{\alpha^\perp!} = \langle R^{TB}(\mathcal{R}^\perp, e_j) \mathcal{R}^\perp, \tilde{e}_i \rangle(Z).$$

From (2.93), (2.94), we still need to obtain the Taylor expansions for  $\theta_j^i(Z^0)$ ,  $(1 \leq i, j \leq 2n - n_0)$  and  $(\partial_k^\perp \theta_j^i)(Z^0)$ ,  $(1 \leq j \leq 2(n - n_0))$ .

By our construction, we know that for  $i$  or  $j \in \{2(n - n_0) + 1, \dots, 2n - n_0\}$ ,

$$(2.95) \quad \tilde{e}_k^\perp(Z^0) = e_k^\perp(Z^0), \quad \theta_j^i(Z^0) = \delta_{i,j}.$$

By [1, (1.21)] (cf. [17, (4.35)]), we know that on  $\mathbb{R}^{2n-2n_0} \times \{0\}$ , for  $i, j \in \{1, \dots, 2(n - n_0)\}$ ,

$$(2.96) \quad \theta_j^i(0) = \delta_{i,j},$$

$$\sum_{|\alpha^0| \geq 1} (|\alpha^0|^2 + |\alpha^0|)((\partial^0)^{\alpha^0} \theta_j^i)(0) \frac{(Z^0)^{\alpha^0}}{\alpha^0!} = \langle R^{TX_G}(\mathcal{R}^0, e_j) \mathcal{R}^0, \tilde{e}_i \rangle(Z^0).$$

while by (0.9), (2.87), and  $[e_i^\perp, e_j^\perp] = 0$ , we get

$$(2.97) \quad (\partial_k^\perp \theta_j^i)(Z^0) = e_k^\perp \langle e_j^0, \tilde{e}_i^0 \rangle(Z^0) = \langle \nabla_{e_k^\perp}^{TB} e_j^0, \tilde{e}_i^0 \rangle(Z^0)$$

$$= \langle \nabla_{e_j^0}^{TB} e_k^\perp, \tilde{e}_i^0 \rangle(Z^0) = -\langle \nabla_{e_j^0}^{TB} \tilde{e}_i^0, e_k^\perp \rangle(Z^0) = -\langle A(e_j^0) \tilde{e}_i^0, e_k^\perp \rangle(Z^0).$$

Let  $R^{TX_G}$ ,  $R^{N_G}$  be the curvatures of  $\nabla^{TX_G}$ ,  $\nabla^{N_G}$ . By (0.9),

$$(2.98) \quad R^{TX_G} + R^{N_G} + A^2 + {}^0\nabla^{TB} A = R^{TB}|_{X_G} \in \Lambda^2(TX_G) \otimes \text{End}(TB).$$

For  $1 \leq j \leq 2(n - n_0)$ ,  $2(n - n_0) + 1 \leq i \leq 2n - n_0$ ,  $i' = i - 2(n - n_0)$ , by  $[e_k^\perp, e_j^0] = 0$ , as in (2.97), we get

$$(2.99) \quad (\partial_k^\perp \theta_j^i)(Z^0) = e_k^\perp \langle e_j^0, \tilde{e}_{i'}^\perp \rangle(Z^0) = \langle \nabla_{e_j^0}^{TB} e_k^\perp, \tilde{e}_{i'}^\perp \rangle(Z^0) = \langle \nabla_{e_j^0}^{N_G} e_k^\perp, e_{i'}^\perp \rangle(Z^0).$$

By [1, Prop. 1.18] (cf. (2.104)) and (2.99), the Taylor expansion of  $(\partial_k^\perp \theta_j^i)(Z^0)$  at 0 to order  $r$  only determines by those of  $R^{N_G}$  to order  $r - 1$ .

Now by (2.87), (2.93)-(2.99) determine the Taylor expansion of  $\theta_j^i(Z)$  to order  $m$  in terms of the Taylor expansion of the curvature coefficients of  $R^{TB}$  to order  $m - 2$  and  $A$  to order  $m - 1$ .

By (2.87), we get Lemma 2.10. □

Let  $dv_{TB}$  be the Riemannian volume form on  $(T_{x_0}B, g^{TB})$ .

Let  $\kappa(Z)$  ( $Z \in \mathbb{R}^{2n-n_0}$ ) be the smooth positive function defined by the equation

$$(2.100) \quad dv_{B_0}(Z) = \kappa(Z) dv_{TB}(Z),$$

with  $\kappa(0) = 1$ .

For  $s \in \mathcal{C}^\infty(\mathbb{R}^{2n-n_0}, \mathbf{E}_{x_0})$  and  $Z \in \mathbb{R}^{2n-n_0}$ , for  $t = \frac{1}{\sqrt{p}}$ , set

$$(2.101) \quad (S_t s)(Z) = s(Z/t), \quad \nabla_t = S_t^{-1} t \kappa^{\frac{1}{2}} \nabla^{E_p, B_0} \kappa^{-\frac{1}{2}} S_t,$$

$$\mathcal{L}_2^t = S_t^{-1} t^2 \kappa^{\frac{1}{2}} \Phi D_p^{X_0, 2} \Phi^{-1} \kappa^{-\frac{1}{2}} S_t.$$

As in (1.18), we denote by  $R^{L_B}$ ,  $R^{E_B}$ ,  $R^{\text{Cliff}_B}$  the curvatures on  $L_B$ ,  $E_B$ ,  $\Lambda(T^{*(0,1)}X)_B$  induced by  $\nabla^L$ ,  $\nabla^E$ ,  $\nabla^{\text{Cliff}}$  on  $X$ .

As in (1.14),  $\tilde{\mu} \in TY$ ,  $\tilde{\mu}^E \in TY \otimes \text{End}(E)$ ,  $\tilde{\mu}^{\text{Cliff}} \in TY \otimes \text{End}(\Lambda(T^{*(0,1)}X))$  are sections induced by  $\mu$ ,  $\mu^E$ ,  $\mu^{\text{Cliff}}$  in (2.15), (2.21).

Denote by  $\nabla_V$  the ordinary differentiation operator on  $T_{x_0}B$  in the direction  $V$ .

Denote by  $(\partial^\alpha R^{L_B})_{x_0}$  the tensor  $(\partial^\alpha R^{L_B})_{x_0}(e_i, e_j) := \partial^\alpha(R^{L_B}(e_i, e_j))_{x_0}$ .

**Theorem 2.11.** *There exist  $\mathcal{A}_{i,j,r}$  ( resp.  $\mathcal{B}_{i,r}$ ,  $\mathcal{C}_r$ ) ( $r \in \mathbb{N}, i, j \in \{1, \dots, 2n - n_0\}$ ) polynomials in  $Z$ , and  $\mathcal{A}_{i,j,r}$  is a monomial in  $Z$  with degree  $r$ , the degree on  $Z$  of  $\mathcal{B}_{i,r}$  (resp.  $\mathcal{C}_r$ ) has the same parity with  $r - 1$  (resp.  $r$ ), with the following properties:*

- the coefficients of  $\mathcal{A}_{i,j,r}$  are polynomials in  $R^{TB}$  (resp.  $A$ ) and their derivatives at  $x_0$  to order  $r - 2$  (resp.  $r - 1$ );
- the coefficients of  $\mathcal{B}_{i,r}$  are polynomials in  $R^{TB}$ ,  $A$ ,  $R^{\text{Cliff}_B}$ ,  $R^{E_B}$ , (resp.  $R^{L_B}$ ) and their derivatives at  $x_0$  to order  $r - 1$  (resp.  $r$ );
- the coefficients of  $\mathcal{C}_r$  are polynomials in  $R^{TB}$ ,  $A$ ,  $R^{\text{Cliff}_B}$ ,  $R^{E_B}$ ,  $\tilde{\mu}^E$ ,  $\tilde{\mu}^{\text{Cliff}}$  (resp.  $r^X$ ,  $\text{Tr}[R^{T^{(1,0)}X}]$ ,  $R^E$ ; resp.  $h$ ,  $R^L$ ,  $R^{L_B}$ ; resp.  $\mu$ ) and their derivatives at  $x_0$  to order  $r - 1$  (resp.  $r - 2$ ; resp.  $r$ ; resp.  $r + 1$ ).
- if we denote by

$$(2.102) \quad \mathcal{O}_r = \mathcal{A}_{i,j,r} \nabla_{e_i} \nabla_{e_j} + \mathcal{B}_{i,r} \nabla_{e_i} + \mathcal{C}_r,$$

$$\mathcal{L}_2^0 = - \sum_{j=1}^{2n-n_0} \left( \nabla_{e_j} + \frac{1}{2} R_{x_0}^{L_B}(\mathcal{R}, e_j) \right)^2 - 2\omega_{d,x_0} - \tau_{x_0} + 4\pi^2 |P^{TY} \mathbf{J}_{x_0} \mathcal{R}|^2,$$

then

$$(2.103) \quad \mathcal{L}_2^t = \mathcal{L}_2^0 + \sum_{r=1}^m t^r \mathcal{O}_r + \mathcal{O}(t^{m+1}).$$

Moreover, there exists  $m' \in \mathbb{N}$  such that for any  $k \in \mathbb{N}$ ,  $t \leq 1$ ,  $|tZ| \leq \varepsilon$ , the derivatives of order  $\leq k$  of the coefficients of the operator  $\mathcal{O}(t^{m+1})$  are dominated by  $Ct^{m+1}(1 + |Z|)^{m'}$ .

*Proof.* Let  $\Gamma^{E_B}$ ,  $\Gamma^{L_B}$  and  $\Gamma^{\text{Cliff}_B}$  be the connection forms of  $\nabla^{E_B}$ ,  $\nabla^{L_B}$  and  $\nabla^{\text{Cliff}_B}$  with respect to any fixed frames for  $E_B$ ,  $L_B$  and  $\Lambda(T^{*(1,0)}X)_B$  which are parallel along the curve  $\gamma_u : [0, 1] \ni u \rightarrow uZ$  under our trivialization on  $B^{T_{x_0}B}(0, \varepsilon)$ . Then  $\Gamma^{E_B}$  is  $\text{End}(\mathbb{C}^{\dim E})$ -valued 1-form on  $\mathbb{R}^{2n-n_0}$  and  $\Gamma^{L_B}$  is 1-form on  $\mathbb{R}^{2n-n_0}$ .

Now for  $\Gamma^\bullet = \Gamma^{E_B}, \Gamma^{L_B}$  or  $\Gamma^{\text{Cliff}_B}$  and  $R^\bullet = R^{E_B}, R^{L_B}$  or  $R^{\text{Cliff}_B}$  respectively, by the definition of our fixed frame and [1, Proposition 1.18] (cf. also [17, (4.45)]), the Taylor coefficients of  $\Gamma^\bullet(e_j)(Z)$  at  $x_0$  to order  $r$  only determines by those of  $R^\bullet$  to order  $r - 1$ , and

$$(2.104) \quad \sum_{|\alpha|=r} (\partial^\alpha \Gamma^\bullet)_{x_0}(e_j) \frac{Z^\alpha}{\alpha!} = \frac{1}{r+1} \sum_{|\alpha|=r-1} (\partial^\alpha R^\bullet)_{x_0}(\mathcal{R}, e_j) \frac{Z^\alpha}{\alpha!}.$$

Especially,

$$(2.105) \quad \Gamma_Z^\bullet(e_j) = \frac{1}{2} R_{x_0}^\bullet(\mathcal{R}, e_j) + \mathcal{O}(|Z|^2).$$

By (2.101), for  $t = 1/\sqrt{p}$ , if  $|Z| \leq \sqrt{p}\varepsilon$ , then

$$(2.106) \quad \nabla_t = \kappa^{\frac{1}{2}}(tZ) \left( \nabla + (t\Gamma^{\text{Cliff}_B} + t\Gamma^{E_B} + \frac{1}{t}\Gamma^{L_B})(tZ) \right) \kappa^{-\frac{1}{2}}(tZ).$$

Moreover, set

$$(2.107) \quad (\nabla_{e_i}^{TB} e_j)(Z) = \Gamma_{ij}^k(Z) e_k, \quad g_{ij}(Z) = g^{TB}(e_i, e_j)(Z) = \theta_i^k \theta_j^k(Z),$$

then  $\Gamma_{ij}^k$  is the connection form of  $\nabla^{TB}$  with respect to the frame  $\{e_i\}$ .

Let  $(g^{ij})$  be the inverse matrix of  $(g_{ij})$ , then

$$(2.108) \quad \Delta^{E_{p,B}} = - \sum_{ij} g^{ij} \left( \nabla_{e_i}^{E_{p,B}} \nabla_{e_j}^{E_{p,B}} - \Gamma_{ij}^k \nabla_{e_k}^{E_{p,B}} \right),$$

and by (1.1), (2.100),

$$(2.109) \quad \begin{aligned} \kappa(Z) &= (\det g_{ij})^{1/2}(Z), \\ \Gamma_{ij}^k &= \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}). \end{aligned}$$

By (2.60), (2.101) and (2.108),

$$(2.110) \quad \begin{aligned} \mathcal{L}_2^t(Z) &= -g^{ij}(tZ) (\nabla_{t,e_i} \nabla_{t,e_j} - t\Gamma_{ij}^k(tZ) \nabla_{t,e_k}) - \langle t\tilde{\mu}^{E_p}, t\tilde{\mu}^{E_p} \rangle_{g^{TY}}(tZ) \\ &\quad - 2\omega_d(tZ) - \tau(tZ) + t^2 \left( \frac{1}{4} r^X + \mathbf{c}(R) - \frac{1}{h} \Delta_{B_0} h \right) (tZ). \end{aligned}$$

By (2.21),

$$(2.111) \quad \langle t\tilde{\mu}^{E_p}, t\tilde{\mu}^{E_p} \rangle_{g^{TY}} = -4\pi^2 \left| \frac{1}{t} \tilde{\mu} \right|_{g^{TY}}^2 + \langle 4\pi\sqrt{-1}\tilde{\mu} + t^2(\tilde{\mu}^{\text{Cliff}} + \tilde{\mu}^E), \tilde{\mu}^{\text{Cliff}} + \tilde{\mu}^E \rangle_{g^{TY}}.$$

By (2.5), (2.15), and  $\tilde{\mu}_{y_0} = 0$ , for  $y_0 \in P$ ,  $\pi(y_0) = x_0$ , we get for  $K \in \mathfrak{g}$ ,

$$(2.112) \quad -\langle \mathbf{J}e_i^H, K^X \rangle_{y_0} = \omega(K^X, e_i^H) = \nabla_{e_i^H}(\mu(K)) = \langle \nabla_{e_i^H}^{TY} \tilde{\mu}, K^X \rangle_{y_0},$$

thus

$$(2.113) \quad |\tilde{\mu}|_{g^{TY}}^2(Z) = |\nabla_{\mathcal{R}}^{TY} \tilde{\mu}|_{g^{TY}}^2 + \mathcal{O}(|Z|^3) = |P^{TY} \mathbf{J}_{x_0} \mathcal{R}|^2 + \mathcal{O}(|Z|^3).$$

By Lemma 2.10, (2.104), (2.106), (2.110) and (2.113), we know that  $\mathcal{L}_2^t$  has the expansion (2.103), in particular, we get the formula  $\mathcal{L}_2^0$  in (2.102).

By (2.98), (2.104) and (2.110), we get the properties on  $\mathcal{A}_{i,j,r}$ ,  $\mathcal{B}_{i,r}$ .

By (2.98), (2.110) and (2.111), we get the properties on  $\mathcal{C}_r$ .

The proof of Theorem 2.11 is complete.  $\square$

**2.7. Uniform estimate on the  $G$ -invariant Bergman kernel.** Recall that the operators  $\mathcal{L}_2^t$ ,  $\nabla_t$  were defined in (2.101), and  $\mathbf{E}_0 = \Lambda(T^{*(0,1)}X_0) \otimes E_0$ . We have trivialized the bundle  $\mathbf{E}_{0,B_0}$  to  $\mathbf{E}_{B,x_0}$  in Section 2.6. We still denote by  $h^{\mathbf{E}_{0,B_0}}$  the metric on the trivial bundle  $\mathbf{E}_{B,x_0}$  on  $\mathbb{R}^{2n-n_0}$  induced by the corresponding metric on  $\mathbf{E}_{0,B_0}$ . Note that  $h^{\mathbf{E}_{0,B_0}}$  is not a constant metric on  $\mathbb{R}^{2n-n_0}$ .

We also denote by  $\langle \cdot, \cdot \rangle_{0,L^2}$  and  $\| \cdot \|_{0,L^2}$  the scalar product and the  $L^2$  norm on  $\mathcal{C}^\infty(T_{x_0}B, \mathbf{E}_{B,x_0})$  induced by  $g^{T_{x_0}B}$ ,  $h^{\mathbf{E}_{0,B_0}}$  as in (1.19).

Let  $\tilde{\mu}_{X_0}, \tilde{\mu}^{E_{0,p}}$  be the  $G$ -invariant sections of  $TY$ ,  $TY \otimes \text{End}(E_{0,p})$  on  $X_0$  induced by  $\mu_{X_0}, \mu^{E_{0,p}}$  as in (1.14).

Let  $\{f_l\}$  be a  $G$ -invariant orthonormal frame of  $TY$  on  $\pi^{-1}(B^B(x_0, \varepsilon))$ , then  $(f_{0,l})_Z = (f_l)_{\varphi_\varepsilon(Z)}$  is a  $G$ -invariant orthonormal frame of  $TY_0$  on  $X_0$ .

**Definition 2.12.** Set

$$(2.114) \quad \mathcal{D}_t = \{ \nabla_{t,e_i}, 1 \leq i \leq 2n - n_0; \frac{1}{t} \langle \tilde{\mu}_{X_0}, f_{0,l} \rangle(tZ), 1 \leq j \leq n_0 \}.$$

For  $k \in \mathbb{N}^*$ , let  $\mathcal{D}_t^k$  be the family of operators acting on  $\mathcal{C}^\infty(T_{x_0}B, \mathbf{E}_{B,x_0})$  which can be written in the form  $Q = Q_1 \cdots Q_k$ ,  $Q_i \in \mathcal{D}_t$ .

For  $s \in \mathcal{C}^\infty(T_{x_0}B, \mathbf{E}_{B,x_0})$ ,  $k \geq 1$ , set

$$(2.115) \quad \begin{aligned} \|s\|_{t,0}^2 &= \int_{\mathbb{R}^{2n-n_0}} |s(Z)|_{h^{\mathbf{E}_0, B_0}(tZ)}^2 dv_{T_{x_0}B}(Z) = t^{-2n+n_0} \|S_t s\|_{0,L^2}^2, \\ \|s\|_{t,k}^2 &= \|s\|_{t,0}^2 + \sum_{l=1}^k \sum_{Q \in \mathcal{D}_t^l} \|Qs\|_{t,0}^2. \end{aligned}$$

We denote by  $\langle s', s \rangle_{t,0}$  the inner product on  $\mathcal{C}^\infty(T_{x_0}B, \mathbf{E}_{B,x_0})$  corresponding to  $\| \cdot \|_{t,0}^2$ .

Let  $H_t^m$  be the Sobolev space of order  $m$  with norm  $\| \cdot \|_{t,m}$ . Let  $H_t^{-1}$  be the Sobolev space of order  $-1$  and let  $\| \cdot \|_{t,-1}$  be the norm on  $H_t^{-1}$  defined by  $\|s\|_{t,-1} = \sup_{0 \neq s' \in H_t^1} |\langle s, s' \rangle_{t,0}| / \|s'\|_{t,1}$ .

If  $A \in \mathcal{L}(H_t^m, H_t^{m'})$  ( $m, m' \in \mathbb{Z}$ ), we denote by  $\|A\|_t^{m,m'}$  the norm of  $A$  with respect to the norms  $\| \cdot \|_{t,m}$  and  $\| \cdot \|_{t,m'}$ .

Then  $\mathcal{L}_2^t$  is a formally self-adjoint elliptic operator with respect to  $\| \cdot \|_{t,0}^2$ , and is a smooth family of operators with respect to the parameter  $x_0 \in X_G$ .

**Theorem 2.13.** *There exist constants  $C_1, C_2, C_3 > 0$  such that for  $t \in ]0, 1]$  and any  $s, s' \in C_0^\infty(\mathbb{R}^{2n-n_0}, \mathbf{E}_{B,x_0})$ ,*

$$(2.116) \quad \begin{aligned} \langle \mathcal{L}_2^t s, s \rangle_{t,0} &\geq C_1 \|s\|_{t,1}^2 - C_2 \|s\|_{t,0}^2, \\ |\langle \mathcal{L}_2^t s, s' \rangle_{t,0}| &\leq C_3 \|s\|_{t,1} \|s'\|_{t,1}. \end{aligned}$$

*Proof.* By (2.83) and our construction for  $L_0, E_0$  on  $X_0$ , we know for  $Z \in T_{x_0}B$ ,  $|Z| > 4\varepsilon$ ,

$$(2.117) \quad \mu^{E_0,p}(K)_{(1,Z)} = p R_{y_0}^L((\mathcal{R}^\perp)^H, K_{y_0}^X).$$

Thus from (2.110) and (2.115),

$$(2.118) \quad \begin{aligned} \langle \mathcal{L}_2^t s, s \rangle_{t,0} &= \|\nabla_t s\|_{t,0}^2 - t^2 \langle \langle \tilde{\mu}^{E_0,p}, \tilde{\mu}^{E_0,p} \rangle_{g_{TY}}(tZ) s, s \rangle_{t,0} \\ &\quad + \left\langle \left( -2S_t^{-1}\omega_d - S_t^{-1}\tau + t^2 S_t^{-1}(\tfrac{1}{4}r^X + \mathbf{c}(R) - \tfrac{1}{h}\Delta_{B_0}h) \right) s, s \right\rangle_{t,0}. \end{aligned}$$

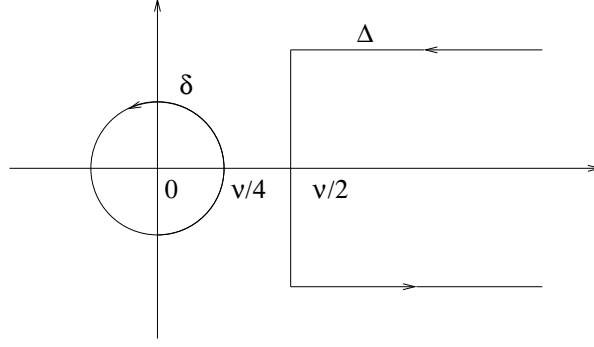
From (2.80), (2.111), (2.117), and our construction on  $\nabla^{E_0}$ ,

$$(2.119) \quad -t^2 \langle \langle \tilde{\mu}^{E_0,p}, \tilde{\mu}^{E_0,p} \rangle_{g_{TY}}(tZ) s, s \rangle_{t,0} \geq 2\pi^2 \sum_{l=1}^{n_0} \left\| \frac{1}{t} \langle \tilde{\mu}_{X_0}, f_{0,l} \rangle(tZ) s \right\|_{t,0}^2 - Ct \|s\|_{t,0}^2.$$

From (2.118) and (2.119), we get (2.116).  $\square$

Recall that  $\nu$  is the constant in (2.23).

Let  $\delta$  be the counterclockwise oriented circle in  $\mathbb{C}$  of center 0 and radius  $\nu/4$ , and let  $\Delta$  be the oriented path in  $\mathbb{C}$  which goes parallel to the real axis from  $+\infty + i$  to  $\frac{\nu}{2} + i$  then parallel to the imaginary axis to  $\frac{\nu}{2} - i$  and the parallel to the real axis to  $+\infty - i$ .



**Theorem 2.14.** *There exist  $t_0 > 0$ ,  $C > 0$  such that for  $t \in ]0, t_0]$ ,  $\lambda \in \delta \cup \Delta$  and  $x_0 \in X_G$ ,  $(\lambda - \mathcal{L}_2^t)^{-1}$  exists and*

$$(2.120) \quad \begin{aligned} \|(\lambda - \mathcal{L}_2^t)^{-1}\|_t^{0,0} &\leq C, \\ \|(\lambda - \mathcal{L}_2^t)^{-1}\|_t^{-1,1} &\leq C(1 + |\lambda|^2). \end{aligned}$$

*Proof.* By (2.23), (2.60) for  $D_p^{X_0}$ , and (2.101), there exists  $t_0 > 0$  such that for  $t \in ]0, t_0]$ ,

$$(2.121) \quad \text{Spec}(\mathcal{L}_2^t) \subset \{0\} \cup [\nu, +\infty[.$$

Thus  $(\lambda - \mathcal{L}_2^t)^{-1}$  exists for  $\lambda \in \delta \cup \Delta$ .

The first inequality of (2.120) is from (2.121).

By (2.116), for  $\lambda_0 \in \mathbb{R}$ ,  $\lambda_0 \leq -2C_2$ ,  $(\lambda_0 - \mathcal{L}_2^t)^{-1}$  exists, and we have  $\|(\lambda_0 - \mathcal{L}_2^t)^{-1}\|_t^{-1,1} \leq \frac{1}{C_1}$ . Now,

$$(2.122) \quad (\lambda - \mathcal{L}_2^t)^{-1} = (\lambda_0 - \mathcal{L}_2^t)^{-1} - (\lambda - \lambda_0)(\lambda - \mathcal{L}_2^t)^{-1}(\lambda_0 - \mathcal{L}_2^t)^{-1}.$$

Thus for  $\lambda \in \delta \cup \Delta$ , from (2.122), we get

$$(2.123) \quad \|(\lambda - \mathcal{L}_2^t)^{-1}\|_t^{-1,0} \leq \frac{1}{C_1} \left(1 + \frac{4}{\nu} |\lambda - \lambda_0|\right).$$

Now we change the last two factors in (2.122), and apply (2.123), we get

$$(2.124) \quad \begin{aligned} \|(\lambda - \mathcal{L}_2^t)^{-1}\|_t^{-1,1} &\leq \frac{1}{C_1} + \frac{|\lambda - \lambda_0|}{C_1^2} \left(1 + \frac{4}{\nu} |\lambda - \lambda_0|\right) \\ &\leq C(1 + |\lambda|^2). \end{aligned}$$

The proof of our Theorem is complete.  $\square$

**Proposition 2.15.** *Take  $m \in \mathbb{N}^*$ . There exists  $C_m > 0$  such that for  $t \in ]0, 1]$ ,  $Q_1, \dots, Q_m \in \mathcal{D}_t \cup \{Z_i\}_{i=1}^{2n-n_0}$  and  $s, s' \in \mathcal{C}_0^\infty(\mathbb{R}^{2n-n_0}, \mathbf{E}_{B,x_0})$ ,*

$$(2.125) \quad \left| \langle [Q_1, [Q_2, \dots, [Q_m, \mathcal{L}_2^t]] \dots ] s, s' \rangle_{t,0} \right| \leq C_m \|s\|_{t,1} \|s'\|_{t,1}.$$

*Proof.* Note that  $[\nabla_{t,e_i}, Z_j] = \delta_{ij}$ . By (2.110), we know that  $[Z_j, \mathcal{L}_2^t]$  verifies (2.125).

Recall that by (2.80) and (2.83),  $(\nabla_{e_i} \langle \tilde{\mu}_{X_0}, f_{0,l} \rangle)(tZ)$  is uniformly bounded with its derivatives for  $t \in [0, 1]$  and

$$(2.126) \quad \nabla_{e_i} \langle \tilde{\mu}_{X_0}, f_{0,l} \rangle = (e_i \langle \tilde{\mu}_{X_0}, f_{0,l} \rangle)_{x_0} = \omega(f_{0,l}, e_i)_{x_0}$$

for  $|Z| \geq 4\varepsilon$ . Thus  $[\frac{1}{t} \langle \tilde{\mu}_{X_0}, f_{0,l} \rangle(tZ), \mathcal{L}_2^t]$  also verifies (2.125).

Note that by (2.101),

$$(2.127) \quad [\nabla_{t,e_i}, \nabla_{t,e_j}] = (R^{L_0, B_0}(tZ) + t^2 R^{\mathbf{E}_0, B_0}(tZ)) (e_i, e_j).$$

Thus from (2.110), (2.126) and (2.127), we know that  $[\nabla_{t,e_k}, \mathcal{L}_2^t]$  has the same structure as  $\mathcal{L}_2^t$  for  $t \in ]0, 1]$ , i.e.  $[\nabla_{t,e_k}, \mathcal{L}_2^t]$  has the type as

$$(2.128) \quad \sum_{ij} a_{ij}(t, tZ) \nabla_{t,e_i} \nabla_{t,e_j} + \sum_i c_i(t, tZ) \nabla_{t,e_i} \\ + \sum_l \left[ c'_l(t, tZ) \frac{1}{t} \langle \tilde{\mu}_{X_0}, f_{0,l} \rangle(tZ) + a \left| \frac{1}{t} \tilde{\mu}_{X_0} \right|_{g^{TY}}^2(tZ) \right] + c(t, tZ),$$

where  $a \in \mathbb{C}$ ;  $a_{ij}(t, Z)$ ,  $c_i(t, Z)$ ,  $c'_j(t, Z)$ ,  $c(t, Z)$  and their derivatives on  $Z$  are uniformly bounded for  $Z \in \mathbb{R}^{2n-n_0}$ ,  $t \in [0, 1]$ ; moreover, they are polynomials in  $t$ . In fact, for  $[\nabla_{t,e_k}, \mathcal{L}_2^t]$ ,  $a = 0$  in (2.128).

Let  $(\nabla_{t,e_i})^*$  be the adjoint of  $\nabla_{t,e_i}$  with respect to  $\langle \cdot, \cdot \rangle_{t,0}$ , then by (2.115),

$$(2.129) \quad (\nabla_{t,e_i})^* = -\nabla_{t,e_i} - t(k^{-1} \nabla_{e_i} k)(tZ),$$

the last term of (2.129) and its derivatives in  $Z$  are uniformly bounded in  $Z \in \mathbb{R}^{2n-n_0}$ ,  $t \in [0, 1]$ .

By (2.128) and (2.129), (2.125) is verified for  $m = 1$ .

By iteration, we know that  $[Q_1, [Q_2, \dots, [Q_m, \mathcal{L}_2^t]] \dots]$  has the same structure (2.128) as  $\mathcal{L}_2^t$ . By (2.129), we get Proposition 2.15.  $\square$

**Theorem 2.16.** *For any  $t \in ]0, t_0]$ ,  $\lambda \in \delta \cup \Delta$ ,  $m \in \mathbb{N}$ , the resolvent  $(\lambda - \mathcal{L}_2^t)^{-1}$  maps  $H_t^m$  into  $H_t^{m+1}$ . Moreover for any  $\alpha \in \mathbb{Z}^{2n-n_0}$ , there exist  $N \in \mathbb{N}$ ,  $C_{\alpha,m} > 0$  such that for  $t \in ]0, t_0]$ ,  $\lambda \in \delta \cup \Delta$ ,  $s \in \mathcal{C}_0^\infty(\mathbb{R}^{2n-n_0}, \mathbf{E}_{B,x_0})$ ,*

$$(2.130) \quad \|Z^\alpha (\lambda - \mathcal{L}_2^t)^{-1} s\|_{t,m+1} \leq C_{\alpha,m} (1 + |\lambda|^2)^N \sum_{\alpha' \leq \alpha} \|Z^{\alpha'} s\|_{t,m}.$$

*Proof.* For  $Q_1, \dots, Q_m \in \mathcal{D}_t$ ,  $Q_{m+1}, \dots, Q_{m+|\alpha|} \in \{Z_i\}_{i=1}^{2n-n_0}$ , we can express  $Q_1 \dots Q_{m+|\alpha|} (\lambda - \mathcal{L}_2^t)^{-1}$  as a linear combination of operators of the type

$$(2.131) \quad [Q_1, [Q_2, \dots [Q_{m'}, (\lambda - \mathcal{L}_2^t)^{-1}]] \dots] Q_{m'+1} \dots Q_{m+|\alpha|}, \quad m' \leq m + |\alpha|.$$

Let  $\mathcal{R}_t$  be the family of operators

$$\mathcal{R}_t = \{[Q_{j_1}, [Q_{j_2}, \dots [Q_{j_l}, \mathcal{L}_2^t]] \dots]\}.$$

Clearly, any commutator  $[Q_1, [Q_2, \dots [Q_{m'}, (\lambda - \mathcal{L}_2^t)^{-1}]] \dots]$  is a linear combination of operators of the form

$$(2.132) \quad (\lambda - \mathcal{L}_2^t)^{-1} R_1 (\lambda - \mathcal{L}_2^t)^{-1} R_2 \dots R_{m'} (\lambda - \mathcal{L}_2^t)^{-1}$$

with  $R_1, \dots, R_{m'} \in \mathcal{R}_t$ .

By Proposition 2.15, the norm  $\| \cdot \|_t^{1,-1}$  of the operators  $R_j \in \mathcal{R}_t$  is uniformly bound by  $C$ .

By Theorem 2.14, we find that there exist  $C > 0$ ,  $N \in \mathbb{N}$  such that the norm  $\| \cdot \|_t^{0,1}$  of operators (2.132) is dominated by  $C(1 + |\lambda|^2)^N$ .  $\square$



Let  $\pi_B : TB \times_B TB \rightarrow B$  be the natural projection from the fiberwise product of  $TB$  on  $B$ .

Let  $e^{-u\mathcal{L}_2^t}(Z, Z')$ ,  $(\mathcal{L}_2^t e^{-u\mathcal{L}_2^t})(Z, Z')$  be the smooth kernels of the operators  $e^{-u\mathcal{L}_2^t}$ ,  $\mathcal{L}_2^t e^{-u\mathcal{L}_2^t}$  with respect to  $dv_{T_{x_0}B}(Z')$ .

Note that  $\mathcal{L}_2^t$  are families of differential operators with coefficients in  $\text{End}(\mathbf{E}_{B,x_0}) = \text{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{B,x_0}$ . Thus we can view  $e^{-u\mathcal{L}_2^t}(Z, Z')$ ,  $(\mathcal{L}_2^t e^{-u\mathcal{L}_2^t})(Z, Z')$  as smooth sections of  $\pi_B^*(\text{End}(\Lambda(T^{*(0,1)}X) \otimes E)_B)$  on  $TB \times_B TB$ .

Let  $\nabla^{\text{End}(\mathbf{E}_B)}$  be the connection on  $\text{End}(\Lambda(T^{*(0,1)}X) \otimes E)_B$  induced by  $\nabla^{\text{Cliff}_B}$  and  $\nabla^{E_B}$ . And  $\nabla^{\text{End}(\mathbf{E}_B)}$ ,  $h^E$  and  $g^{TX}$  induce naturally a  $\mathcal{C}^m$ -norm for the parameter  $x_0 \in X_G$ .

As in Introduction, for  $Z \in T_{x_0}B$ , we will write  $Z = Z^0 + Z^\perp$ , with  $Z^0 \in T_{x_0}X_G$ ,  $Z^\perp \in N_{G,x_0}$ .

**Theorem 2.17.** *There exists  $C'' > 0$  such that for any  $m, m', m'', r \in \mathbb{N}$ ,  $u_0 > 0$ , there exists  $C > 0$  such that for  $t \in ]0, t_0]$ ,  $u \geq u_0$ ,  $Z, Z' \in T_{x_0}B$ ,*

$$\begin{aligned}
(2.133) \quad & \sup_{|\alpha|+|\alpha'|\leq m} (1 + |Z^\perp| + |Z'^\perp|)^{m''} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \frac{\partial^r}{\partial t^r} e^{-u\mathcal{L}_2^t}(Z, Z') \right|_{\mathcal{C}^{m'}(X_G)} \\
& \leq C(1 + |Z^0| + |Z'^0|)^{2(n+r+m'+1)+m} \exp\left(\frac{1}{2}\nu u - \frac{2C''}{u}|Z - Z'|^2\right), \\
& \sup_{|\alpha|+|\alpha'|\leq m} (1 + |Z^\perp| + |Z'^\perp|)^{m''} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \frac{\partial^r}{\partial t^r} (\mathcal{L}_2^t e^{-u\mathcal{L}_2^t})(Z, Z') \right|_{\mathcal{C}^{m'}(X_G)} \\
& \leq C(1 + |Z^0| + |Z'^0|)^{2(n+r+m'+1)+m} \exp\left(-\frac{1}{4}\nu u - \frac{2C''}{u}|Z - Z'|^2\right),
\end{aligned}$$

where  $\mathcal{C}^{m'}(X_G)$  is the  $\mathcal{C}^{m'}$  norm for the parameter  $x_0 \in X_G$ .

*Proof.* By (2.121), for any  $k \in \mathbb{N}^*$ ,

$$\begin{aligned}
(2.134) \quad & e^{-u\mathcal{L}_2^t} = \frac{(-1)^{k-1}(k-1)!}{2\pi i u^{k-1}} \int_{\delta \cup \Delta} e^{-u\lambda} (\lambda - \mathcal{L}_2^t)^{-k} d\lambda, \\
& \mathcal{L}_2^t e^{-u\mathcal{L}_2^t} = \frac{(-1)^{k-1}(k-1)!}{2\pi i u^{k-1}} \int_{\Delta} e^{-u\lambda} \left[ \lambda (\lambda - \mathcal{L}_2^t)^{-k} - (\lambda - \mathcal{L}_2^t)^{-k+1} \right] d\lambda.
\end{aligned}$$

From Theorem 2.16, we deduce that if  $Q \in \cup_{l=1}^m \mathcal{D}_t^l$ , there are  $N \in \mathbb{N}$ ,  $C_m > 0$  such that for any  $\lambda \in \delta \cup \Delta$ ,

$$(2.135) \quad \|Q(\lambda - \mathcal{L}_2^t)^{-m}\|_t^{0,0} \leq C_m(1 + |\lambda|^2)^N.$$

Recall that  $\mathcal{L}_2^t$  is self-adjoint with respect to  $\|\cdot\|_{t,0}$ . After taking the adjoint of (2.135), we get

$$(2.136) \quad \|(\lambda - \mathcal{L}_2^t)^{-m}Q\|_t^{0,0} \leq C_m(1 + |\lambda|^2)^N.$$

From (2.134), (2.135) and (2.136), we get if  $Q, Q' \in \cup_{l=1}^m \mathcal{D}_t^l$ ,

$$\begin{aligned}
(2.137) \quad & \|Qe^{-u\mathcal{L}_2^t}Q'\|_t^{0,0} \leq C_m e^{\frac{1}{4}\nu u}, \\
& \|Q(\mathcal{L}_2^t e^{-u\mathcal{L}_2^t})Q'\|_t^{0,0} \leq C_m e^{-\frac{1}{2}\nu u}.
\end{aligned}$$

Let  $|\cdot|_m$  be the usual Sobolev norm on  $\mathcal{C}^\infty(\mathbb{R}^{2n-n_0}, \mathbf{E}_{B,x_0})$  induced by  $h^{\mathbf{E}_{B,x_0}} = h^{(\Lambda(T^{*(0,1)}X) \otimes E)_{B,x_0}}$  and the volume form  $dv_{T_{x_0}B}(Z)$  as in (2.115).

Observe that by (2.106), (2.115), there exists  $C > 0$  such that for  $s \in \mathcal{C}^\infty(T_{x_0}B, \mathbf{E}_{B,x_0})$ ,  $\text{supp}(s) \subset B^{T_{x_0}B}(0, q)$ ,  $m \geq 0$ ,

$$(2.138) \quad \frac{1}{C}(1+q)^{-m}\|s\|_{t,m} \leq |s|_m \leq C(1+q)^m\|s\|_{t,m}.$$

Now (2.137), (2.138) together with Sobolev's inequalities imply that if  $Q, Q' \in \cup_{l=1}^m \mathcal{D}_t^l$ , for  $\mathcal{K}_u(\mathcal{L}_2^t) = e^{-\frac{1}{4}\nu u} e^{-u\mathcal{L}_2^t}$  or  $e^{\frac{1}{2}\nu u} \mathcal{L}_2^t e^{-u\mathcal{L}_2^t}$ , we have

$$(2.139) \quad \sup_{|Z|, |Z'| \leq q} |Q_Z Q'_{Z'} \mathcal{K}_u(\mathcal{L}_2^t)(Z, Z')| \leq C(1+q)^{2n+2}.$$

By (2.80), (2.81) and (2.83),

$$(2.140) \quad \sum_{l=1}^{n_0} \left| \frac{1}{t} \langle \tilde{\mu}_{X_0}, f_{0,l} \rangle(tZ) \right|^2 = \left| \frac{1}{t} \tilde{\mu}_{X_0} \right|_{g_{TY}}^2(tZ) \geq C|Z^\perp|^2.$$

Thus by (2.106), (2.139), (2.140), we derive (2.133) with the exponentials  $e^{\frac{1}{4}\nu u}$ ,  $e^{-\frac{1}{2}\nu u}$  for the case when  $r = m' = 0$  and  $C'' = 0$ , i.e.

$$(2.141) \quad \sup_{|\alpha|+|\alpha'| \leq m} (1 + |Z^\perp| + |Z'^\perp|)^{m''} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \mathcal{K}_u(\mathcal{L}_2^t)(Z, Z') \right| \leq C(1 + |Z^0| + |Z'^0|)^{2n+m+2}.$$

To obtain (2.133) in general, we proceed as in the proof of [4, Theorem 11.14].

Note that the function  $f$  is defined in (2.28). For  $\varrho > 1$ , put

$$(2.142) \quad K_{u,\varrho}(a) = \int_{-\infty}^{+\infty} \exp(iv\sqrt{2u}a) \exp\left(-\frac{v^2}{2}\right) \left(1 - f\left(\frac{1}{\varrho}\sqrt{2uv}\right)\right) \frac{dv}{\sqrt{2\pi}}.$$

Then there exist  $C', C_1 > 0$  such that for any  $c > 0$ ,  $m, m' \in \mathbb{N}$ , there is  $C > 0$  such that for  $u \geq u_0$ ,  $a \in \mathbb{C}$ ,  $|\text{Im}(a)| \leq c$ , we have

$$(2.143) \quad |a|^m |K_{u,\varrho}^{(m')}(a)| \leq C \exp\left(C'c^2u - \frac{C_1}{u}\varrho^2\right).$$

For any  $c > 0$ , let  $V_c$  be the images of  $\{\lambda \in \mathbb{C}, |\text{Im}(\lambda)| \leq c\}$  by the map  $\lambda \rightarrow \lambda^2$ . Then

$$V_c = \{\lambda \in \mathbb{C}, \text{Re}(\lambda) \geq \frac{1}{4c^2} \text{Im}(\lambda)^2 - c^2\},$$

and  $\delta \cup \Delta \subset V_c$  for  $c$  large enough.

Let  $\tilde{K}_{u,\varrho}$  be the holomorphic function such that  $\tilde{K}_{u,\varrho}(a^2) = K_{u,\varrho}(a)$ .

By (2.143), for  $\lambda \in V_c$ ,

$$(2.144) \quad |\lambda|^m |\tilde{K}_{u,\varrho}^{(m')}(\lambda)| \leq C \exp\left(C'c^2u - \frac{C_1}{u}\varrho^2\right).$$

Using finite propagation speed of solutions of hyperbolic equations and (2.142), we find that there exists a fixed constant (which depends on  $\varepsilon$ )  $c' > 0$  such that

$$(2.145) \quad \tilde{K}_{u,\varrho}(\mathcal{L}_2^t)(Z, Z') = e^{-u\mathcal{L}_2^t}(Z, Z') \quad \text{if } |Z - Z'| \geq c'\varrho.$$

By (2.144), we see that given  $k \in \mathbb{N}$ , there is a unique holomorphic function  $\tilde{K}_{u,\varrho,k}(\lambda)$  defined on a neighborhood of  $V_c$  such that it verifies the same estimates as  $\tilde{K}_{u,\varrho}$  in (2.144) and  $\tilde{K}_{u,\varrho,k}(\lambda) \rightarrow 0$  as  $\lambda \rightarrow +\infty$ ; moreover

$$(2.146) \quad \tilde{K}_{u,\varrho,k}^{(k-1)}(\lambda)/(k-1)! = \tilde{K}_{u,\varrho}(\lambda).$$

Thus as in (2.134),

$$(2.147) \quad \begin{aligned} \tilde{K}_{u,\varrho}(\mathcal{L}_2^t) &= \frac{1}{2\pi i} \int_{\delta \cup \Delta} \tilde{K}_{u,\varrho,k}(\lambda)(\lambda - \mathcal{L}_2^t)^{-k} d\lambda, \\ \mathcal{L}_2^t \tilde{K}_{u,\varrho}(\mathcal{L}_2^t) &= \frac{1}{2\pi i} \int_{\Delta} \tilde{K}_{u,\varrho,k}(\lambda) \left[ \lambda(\lambda - \mathcal{L}_2^t)^{-k} - (\lambda - \mathcal{L}_2^t)^{-k+1} \right] d\lambda. \end{aligned}$$

By (2.135), (2.136) and by proceeding as in (2.137)-(2.139), we find that for  $\mathbf{K}_u(a) = \tilde{K}_{u,\varrho}(a)$  or  $a\tilde{K}_{u,\varrho}(a)$ , for  $|Z|, |Z'| \leq q$ ,

$$(2.148) \quad \sup_{|\alpha|+|\alpha'| \leq m} (1 + |Z^\perp| + |Z'^\perp|)^{2n+m+m''+2} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \mathbf{K}_u(\mathcal{L}_2^t)(Z, Z') \right| \leq C(1+q)^{2n+2+m} \exp(C'c^2u - \frac{C_1}{u}\varrho^2).$$

Setting  $\varrho \in \mathbb{N}^*$ ,  $|\varrho - \frac{1}{c'}|Z - Z'| < 1$  in (2.148), we get for  $\alpha, \alpha'$  verifying  $|\alpha| + |\alpha'| \leq m$ ,

$$(2.149) \quad (1 + |Z^\perp| + |Z'^\perp|)^{m''} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \mathbf{K}_u(\mathcal{L}_2^t)(Z, Z') \right| \leq C(1 + |Z^0| + |Z'^0|)^{2n+m+2} \exp(C'c^2u - \frac{C_1}{2c'^2u}|Z - Z'|^2).$$

Take  $\delta_1 = \frac{C'c^2 + \frac{1}{2}\nu}{C'c^2 + \frac{1}{2}\nu}$ , from (2.141) $^{\delta_1} \times$  (2.149) $^{1-\delta_1}$  and (2.145), we get (2.133) for  $r = m' = 0$ .

To get (2.133) for  $r \geq 1$ , note that from (2.134), for  $k \geq 1$

$$(2.150) \quad \frac{\partial^r}{\partial t^r} e^{-u\mathcal{L}_2^t} = \frac{(-1)^{k-1}(k-1)!}{2\pi i u^{k-1}} \int_{\delta \cup \Delta} e^{-u\lambda} \frac{\partial^r}{\partial t^r} (\lambda - \mathcal{L}_2^t)^{-k} d\lambda.$$

We have the similar equation for  $\frac{\partial^r}{\partial t^r} (\mathcal{L}_2^t e^{-u\mathcal{L}_2^t})$ .

Set

$$(2.151) \quad I_{k,r} = \left\{ (\mathbf{k}, \mathbf{r}) = (k_i, r_i) \mid \sum_{i=0}^j k_i = k + j, \sum_{i=1}^j r_i = r, k_i, r_i \in \mathbb{N}^* \right\}.$$

Then there exist  $a_{\mathbf{r}}^{\mathbf{k}} \in \mathbb{R}$  such that

$$(2.152) \quad \begin{aligned} A_{\mathbf{r}}^{\mathbf{k}}(\lambda, t) &= (\lambda - \mathcal{L}_2^t)^{-k_0} \frac{\partial^{r_1} \mathcal{L}_2^t}{\partial t^{r_1}} (\lambda - \mathcal{L}_2^t)^{-k_1} \dots \frac{\partial^{r_j} \mathcal{L}_2^t}{\partial t^{r_j}} (\lambda - \mathcal{L}_2^t)^{-k_j}, \\ \frac{\partial^r}{\partial t^r} (\lambda - \mathcal{L}_2^t)^{-k} &= \sum_{(\mathbf{k}, \mathbf{r}) \in I_{k,r}} a_{\mathbf{r}}^{\mathbf{k}} A_{\mathbf{r}}^{\mathbf{k}}(\lambda, t). \end{aligned}$$

We claim that  $A_{\mathbf{r}}^{\mathbf{k}}(\lambda, t)$  is well defined and for any  $m \in \mathbb{N}$ ,  $k > 2(m + r + 1)$ ,  $Q, Q' \in \cup_{l=1}^m \mathcal{D}_t^l$ , there exist  $C > 0$ ,  $N \in \mathbb{N}$  such that for  $\lambda \in \delta \cup \Delta$ ,

$$(2.153) \quad \|QA_{\mathbf{r}}^{\mathbf{k}}(\lambda, t)Q's\|_{t,0} \leq C(1 + |\lambda|)^N \sum_{|\beta| \leq 2r} \|Z^\beta s\|_{t,0}.$$

In fact, by (2.110),  $\frac{\partial^r}{\partial t^r} \mathcal{L}_2^t$  is a combination of  $\frac{\partial^{r_1}}{\partial t^{r_1}}(g^{ij}(tZ))$ ,  $(\frac{\partial^{r_2}}{\partial t^{r_2}} \nabla_{t,e_i})$ ,  $\frac{\partial^{r_3}}{\partial t^{r_3}}(q(tZ))$ ,  $\frac{\partial^{r_4}}{\partial t^{r_4}}(t\langle \tilde{\mu}^{E_{0,p}}, f_{0,l}(tZ) \rangle)$ , where  $q$  runs over the functions  $r^X$ , etc., appearing in (2.110).

Now  $\frac{\partial^{r_1}}{\partial t^{r_1}}(q(tZ))$  (resp.  $\frac{\partial^{r_1}}{\partial t^{r_1}}(t\langle \tilde{\mu}^{E_{0,p}}, f_{0,l}(tZ) \rangle)$ ,  $\frac{\partial^{r_1}}{\partial t^{r_1}} \nabla_{t,e_i}$  ( $r_1 \geq 1$ )) are functions of the type as  $q'(tZ)Z^\beta$ ,  $|\beta| \leq r_1$  (resp.  $r_1 + 1$ ) (where  $q'$ , as  $q$ , runs over the functions  $r^X$ , etc., appearing in (2.110)), with  $q'(Z)$  and its derivatives on  $Z$  being bounded smooth functions on  $Z$ .

Let  $\mathcal{R}'_t$  be the family of operators of the type

$$\mathcal{R}'_t = \{[f_{j_1}Q_{j_1}, [f_{j_2}Q_{j_2}, \dots [f_{j_l}Q_{j_l}, \mathcal{L}_2^t]] \dots]\}$$

with  $f_{j_i}$  smooth bounded (with its derivatives) functions and  $Q_{j_i} \in \mathcal{D}_t \cup \{Z_j\}_{j=1}^{2n-n_0}$ .

Now for the operator  $A_{\mathbf{r}}^{\mathbf{k}}(\lambda, t)Q'$ , we will move first all the term  $Z^\beta$  in  $d'(tZ)Z^\beta$  as above to the right hand side of this operator, to do so, we always use the commutator trick, i.e., each time, we consider only the commutation for  $Z_i$ , not for  $Z^\beta$  with  $|\beta| > 1$ .

Then  $A_{\mathbf{r}}^{\mathbf{k}}(\lambda, t)Q'$  is as the form  $\sum_{|\beta| \leq 2r} L_\beta^t Q''_\beta Z^\beta$ , and  $Q''_\beta$  is obtained from  $Q'$  and its commutation with  $Z^\beta$ .

Now we move all the terms  $\nabla_{t,e_i}$ ,  $\langle \frac{1}{t} \tilde{\mu}, f_{0,l} \rangle(tZ)$  in  $\frac{\partial^{r_j} \mathcal{L}_2^t}{\partial t^{r_j}}$  to the right hand side of the operator  $L_\beta^t$ .

Then as in the proof of Theorem 2.16, we get finally that  $QA_{\mathbf{r}}^{\mathbf{k}}(\lambda, t)Q'$  is as the form  $\sum_\beta \mathcal{L}_\beta^t Z^\beta$  where  $\mathcal{L}_\beta^t$  is a linear combination of operators of the form

$$(2.154) \quad Q(\lambda - \mathcal{L}_2^t)^{-k'_0} R_1(\lambda - \mathcal{L}_2^t)^{-k'_1} R_2 \dots R_{l'}(\lambda - \mathcal{L}_2^t)^{-k'_{l'}} Q''' Q'',$$

with  $R_1, \dots, R_{l'} \in \mathcal{R}'_t$ ,  $Q''' \in \cup_{l=1}^{2r} \mathcal{D}_t^l$ ,  $Q'' \in \cup_{l=1}^m \mathcal{D}_t^l$ ,  $|\beta| \leq 2r$ , and  $Q''$  is obtained from  $Q'$  and its commutation with  $Z^\beta$ .

By the argument as in (2.135) and (2.136), as  $k > 2(m + r + 1)$ , we can split the above operator to two parts

$$\begin{aligned} & Q(\lambda - \mathcal{L}_2^t)^{-k'_0} R_1(\lambda - \mathcal{L}_2^t)^{-k'_1} R_2 \dots R_i(\lambda - \mathcal{L}_2^t)^{-k'_i}; \\ & (\lambda - \mathcal{L}_2^t)^{-(k'_i - k''_i)} \dots R_{l'}(\lambda - \mathcal{L}_2^t)^{-k'_{l'}} Q''' Q'', \end{aligned}$$

and the  $\|\cdot\|_t^{0,0}$ -norm of each part is bounded by  $C(1 + |\lambda|^2)^N$ .

Thus the proof of (2.153) is complete.

By (2.150), (2.152) and (2.153), we get the similar estimate (2.141), (2.149) for  $\frac{\partial^r}{\partial t^r} e^{-u\mathcal{L}_2^t}$ ,  $\frac{\partial^r}{\partial t^r} (\mathcal{L}_2^t e^{-u\mathcal{L}_2^t})$  with the exponential  $2n + m + 2r + 2$  instead of  $2n + m + 2$  therein.

Thus we get (2.133) for  $m' = 0$ .

Finally, for  $U \in TX_G$  a vector on  $X_G$ ,

$$(2.155) \quad \nabla_U^{\pi^* \text{End}(\mathbf{E}_B)} e^{-u\mathcal{L}_2^t} = \frac{(-1)^{k-1}(k-1)!}{2\pi i u^{k-1}} \int_{\delta \cup \Delta} e^{-u\lambda} \nabla_U^{\pi^* \text{End}(\mathbf{E}_B)} (\lambda - \mathcal{L}_2^t)^{-k} d\lambda.$$

Now, by using the similar formula (2.152) for  $\nabla_U^{\pi^* \text{End}(\mathbf{E}_B)}(\lambda - \mathcal{L}_2^t)^{-k}$  by replacing  $\frac{\partial^{r_1} \mathcal{L}_2^t}{\partial t^{r_1}}$  by  $\nabla_U^{\pi^* \text{End}(\mathbf{E}_B)} \mathcal{L}_2^t$ , and remark that  $\nabla_U^{\pi^* \text{End}(\mathbf{E}_B)} \mathcal{L}_2^t$  is a differential operator on  $T_{x_0} B$  with the same structure as  $\mathcal{L}_2^t$ .

Then by the above argument, we get (2.133) for  $m' \geq 1$ .  $\square$

Let  $P_{0,t}$  be the orthogonal projection from  $\mathcal{C}^\infty(T_{x_0} B, \mathbf{E}_{B,x_0})$  to the kernel of  $\mathcal{L}_2^t$  with respect to  $\langle \cdot, \cdot \rangle_{t,0}$ . Set

$$(2.156) \quad F_u(\mathcal{L}_2^t) = \frac{1}{2\pi i} \int_{\Delta} e^{-u\lambda} (\lambda - \mathcal{L}_2^t)^{-1} d\lambda.$$

By (2.121),

$$(2.157) \quad F_u(\mathcal{L}_2^t) = e^{-u\mathcal{L}_2^t} - P_{0,t} = \int_u^{+\infty} \mathcal{L}_2^t e^{-u_1 \mathcal{L}_2^t} du_1.$$

Let  $P_{0,t}(Z, Z')$ ,  $F_u(\mathcal{L}_2^t)(Z, Z')$  be the smooth kernels of  $P_{0,t}$ ,  $F_u(\mathcal{L}_2^t)$  with respect to  $dv_{T_{x_0} B}(Z')$ .

**Corollary 2.18.** *With the notation in Theorem 2.17,*

$$(2.158) \quad \sup_{|\alpha|+|\alpha'| \leq m} (1 + |Z^\perp| + |Z'^\perp|)^{m'} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \frac{\partial^r}{\partial t^r} F_u(\mathcal{L}_2^t)(Z, Z') \right|_{\mathcal{C}^{m'}(P)} \\ \leq C(1 + |Z^0| + |Z'^0|)^{2n+m+2m'+2r+2} \exp\left(-\frac{1}{8}\nu u - \sqrt{C''\nu}|Z - Z'|\right).$$

*Proof.* Note that  $\frac{1}{8}\nu u + \frac{2C''}{u}|Z - Z'|^2 \geq \sqrt{C''\nu}|Z - Z'|$ , thus

$$(2.159) \quad \int_u^{+\infty} e^{-\frac{1}{4}\nu u_1 - \frac{2C''}{u_1}|Z - Z'|^2} du_1 \leq e^{-\sqrt{C''\nu}|Z - Z'|} \int_u^{+\infty} e^{-\frac{1}{8}\nu u_1} du_1 \\ = \frac{8}{\nu} e^{-\frac{1}{8}\nu u - \sqrt{C''\nu}|Z - Z'|}.$$

By (2.133), (2.157) and (2.159), we get (2.158).  $\square$

For  $k$  large enough, set

$$(2.160) \quad F_{r,u} = \frac{(-1)^{k-1}(k-1)!}{2\pi i r! u^{k-1}} \int_{\Delta} e^{-u\lambda} \sum_{(\mathbf{k}, \mathbf{r}) \in I_{k,r}} a_{\mathbf{r}}^{\mathbf{k}} A_{\mathbf{r}}^{\mathbf{k}}(\lambda, 0) d\lambda, \\ J_{r,u} = \frac{(-1)^{k-1}(k-1)!}{2\pi i r! u^{k-1}} \int_{\delta \cup \Delta} e^{-u\lambda} \sum_{(\mathbf{k}, \mathbf{r}) \in I_{k,r}} a_{\mathbf{r}}^{\mathbf{k}} A_{\mathbf{r}}^{\mathbf{k}}(\lambda, 0) d\lambda, \\ F_{r,u,t} = \frac{1}{r!} \frac{\partial^r}{\partial t^r} F_u(\mathcal{L}_2^t) - F_{r,u}, \quad J_{r,u,t} = \frac{1}{r!} \frac{\partial^r}{\partial t^r} e^{-u\mathcal{L}_2^t} - J_{r,u}.$$

Certainly, as  $t \rightarrow 0$ , the limit of  $\| \cdot \|_{t,m}$  exists, and we denote it by  $\| \cdot \|_{0,m}$ .

**Theorem 2.19.** *For any  $r \geq 0$ ,  $k > 0$ , there exist  $C > 0$ ,  $N \in \mathbb{N}$  such that for  $t \in [0, t_0]$ ,  $\lambda \in \delta \cup \Delta$ ,*

$$(2.161) \quad \left\| \left( \frac{\partial^r \mathcal{L}_2^t}{\partial t^r} - \frac{\partial^r \mathcal{L}_2^t}{\partial t^r} \Big|_{t=0} \right) s \right\|_{t,-1} \leq Ct \sum_{|\alpha| \leq r+3} \|Z^\alpha s\|_{0,1},$$

$$\left\| \left( \frac{\partial^r}{\partial t^r} (\lambda - \mathcal{L}_2^t)^{-k} - \sum_{(\mathbf{k}, \mathbf{r}) \in I_{k,r}} a_{\mathbf{r}}^{\mathbf{k}} A_{\mathbf{r}}^{\mathbf{k}}(\lambda, 0) \right) s \right\|_{0,0} \leq Ct(1 + |\lambda|^2)^N \sum_{|\alpha| \leq 4r+3} \|Z^\alpha s\|_{0,0}.$$

*Proof.* Note that by (2.106), (2.115), for  $t \in [0, 1]$ ,  $k \geq 1$

$$(2.162) \quad \|s\|_{t,0} \leq C\|s\|_{0,0}, \quad \|s\|_{t,k} \leq C \sum_{|\alpha| \leq k} \|Z^\alpha s\|_{0,k}.$$

An application of Taylor expansion for (2.110) leads to the following equation, if  $s, s'$  have compact support,

$$(2.163) \quad \left| \left\langle \left( \frac{\partial^r \mathcal{L}_2^t}{\partial t^r} - \frac{\partial^r \mathcal{L}_2^t}{\partial t^r} \Big|_{t=0} \right) s, s' \right\rangle_{0,0} \right| \leq Ct \|s'\|_{t,1} \sum_{|\alpha| \leq r+3} \|Z^\alpha s\|_{0,1}.$$

Thus we get the first inequality of (2.161).

Note that

$$(2.164) \quad (\lambda - \mathcal{L}_2^t)^{-1} - (\lambda - \mathcal{L}_2^0)^{-1} = (\lambda - \mathcal{L}_2^t)^{-1} (\mathcal{L}_2^t - \mathcal{L}_2^0) (\lambda - \mathcal{L}_2^0)^{-1}.$$

Now from (2.120), (2.163) and (2.164),

$$(2.165) \quad \left\| ((\lambda - \mathcal{L}_2^t)^{-1} - (\lambda - \mathcal{L}_2^0)^{-1}) s \right\|_{0,0} \leq Ct(1 + |\lambda|^4) \sum_{|\alpha| \leq 3} \|Z^\alpha s\|_{0,0}.$$

After taking the limit, we know that Theorems 2.14-2.16 still hold for  $t = 0$ .

Note that  $\nabla_{0,e_j} = \nabla_{e_j} + \frac{1}{2} R_{x_0}^{L_B}(\mathcal{R}, e_j)$  by (2.106).

If we denote by  $\mathcal{L}_{\lambda,t} = \lambda - \mathcal{L}_2^t$ , then

$$(2.166) \quad A_{\mathbf{r}}^{\mathbf{k}}(\lambda, t) - A_{\mathbf{r}}^{\mathbf{k}}(\lambda, 0) = \sum_{i=1}^j \mathcal{L}_{\lambda,t}^{-k_0} \cdots \left( \frac{\partial^{r_i} \mathcal{L}_2^t}{\partial t^{r_i}} - \frac{\partial^{r_i} \mathcal{L}_2^t}{\partial t^{r_i}} \Big|_{t=0} \right) \mathcal{L}_{\lambda,0}^{-k_i} \cdots \mathcal{L}_{\lambda,0}^{-k_j}$$

$$+ \sum_{i=0}^j \mathcal{L}_{\lambda,t}^{-k_0} \cdots (\mathcal{L}_{\lambda,t}^{-k_i} - \mathcal{L}_{\lambda,0}^{-k_i}) \left( \frac{\partial^{r_{i+1}} \mathcal{L}_2^t}{\partial t^{r_{i+1}}} \Big|_{t=0} \right) \cdots \mathcal{L}_{\lambda,0}^{-k_j}.$$

Now from the first inequality of (2.161), (2.120), (2.152), (2.165) and (2.166), we get (2.161).  $\square$

**Theorem 2.20.** *There exist  $C > 0$ ,  $N \in \mathbb{N}$  such that for  $t \in ]0, t_0]$ ,  $u \geq u_0$ ,  $q \in \mathbb{N}$ ,  $Z, Z' \in T_{x_0} B$ ,  $|Z|, |Z'| \leq q$ ,*

$$(2.167) \quad \left| F_{r,u,t}(Z, Z') \right| \leq Ct^{\frac{1}{2(2n-n_0+1)}} (1+q)^N e^{-\frac{1}{8}\nu u},$$

$$\left| J_{r,u,t}(Z, Z') \right| \leq Ct^{\frac{1}{2(2n-n_0+1)}} (1+q)^N e^{\frac{1}{2}\nu u}.$$

*Proof.* Let  $J_{x_0,q}^0$  be the vector space of square integrable sections of  $\mathbf{E}_{B,x_0}$  over  $\{Z \in T_{x_0}B, |Z| \leq q+1\}$ .

If  $s \in J_{x_0,q}^0$ , put  $\|s\|_{(q)}^2 = \int_{|Z| \leq q+1} |s|_{\mathbf{E}_{B,x_0}}^2 dv_{TB}(Z)$ . Let  $\|A\|_{(q)}$  be the operator norm of  $A \in \mathcal{L}(J_{x_0,q}^0)$  with respect to  $\|\cdot\|_{(q)}$ .

By (2.150), (2.160) and (2.161), we get: there exist  $C > 0, N \in \mathbb{N}$  such that for  $t \in ]0, t_0], u \geq u_0$ ,

$$(2.168) \quad \begin{aligned} \|F_{r,u,t}\|_{(q)} &\leq Ct(1+q)^N e^{-\frac{1}{2}\nu u}, \\ \|J_{r,u,t}\|_{(q)} &\leq Ct(1+q)^N e^{\frac{1}{4}\nu u}. \end{aligned}$$

Let  $\phi : \mathbb{R} \rightarrow [0, 1]$  be a smooth function with compact support, equal 1 near 0, such that  $\int_{T_{x_0}B} \phi(Z) dv_{T_{x_0}B}(Z) = 1$ .

Take  $\varsigma \in ]0, 1]$ .

By the proof of Theorem 2.17,  $F_{r,u}$  verifies the similar inequality as in (2.158). Thus by (2.158), there exists  $C > 0$  such that if  $|Z|, |Z'| \leq q, U, U' \in \mathbf{E}_{B,x_0}$ ,

$$(2.169) \quad \left| \langle F_{r,u,t}(Z, Z')U, U' \rangle - \int_{T_{x_0}B \times T_{x_0}B} \langle F_{r,u,t}(Z - W, Z' - W')U, U' \rangle \frac{1}{\varsigma^{4n-2n_0}} \phi(W/\varsigma) \phi(W'/\varsigma) dv_{T_{x_0}B}(W) dv_{T_{x_0}B}(W') \right| \leq C\varsigma(1+q)^N e^{-\frac{1}{8}\nu u} |U||U'|.$$

On the other hand, by (2.168),

$$(2.170) \quad \left| \int_{T_{x_0}B \times T_{x_0}B} \langle F_{r,u,t}(Z - W, Z' - W')U, U' \rangle \frac{1}{\varsigma^{4n-2n_0}} \phi(W/\varsigma) \phi(W'/\varsigma) dv_{T_{x_0}B}(W) dv_{T_{x_0}B}(W') \right| \leq Ct \frac{1}{\varsigma^{2n-n_0}} (1+q)^N e^{-\frac{1}{2}\nu u} |U||U'|.$$

By taking  $\varsigma = t^{1/2(2n-n_0+1)}$ , we get (2.167).

In the same way, we get (2.167) for  $J_{r,u,t}$ .  $\square$

**Theorem 2.21.** *There exists  $C'' > 0$  such that for any  $k, m, m', m'' \in \mathbb{N}$ , there exist  $N \in \mathbb{N}, C > 0$  such that if  $t \in ]0, t_0], u \geq u_0, Z, Z' \in T_{x_0}^H U, \alpha, \alpha' \in \mathbb{Z}^{2n-n_0}, |\alpha| + |\alpha'| \leq m$ ,*

$$(2.171) \quad \begin{aligned} (1 + |Z^\perp| + |Z'^\perp|)^{m''} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left( F_u(\mathcal{L}_2^t) - \sum_{r=0}^k F_{r,u} t^r \right) (Z, Z') \right|_{\mathcal{C}^{m'}(X_G)} \\ \leq Ct^{k+1} (1 + |Z^0| + |Z'^0|)^{2(n+k+m'+2)+m} \exp\left(-\frac{1}{8}\nu u - \sqrt{C''}\nu|Z - Z'|\right), \\ (1 + |Z^\perp| + |Z'^\perp|)^{m''} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left( e^{-u\mathcal{L}_2^t} - \sum_{r=0}^k J_{r,u} t^r \right) (Z, Z') \right|_{\mathcal{C}^{m'}(X_G)} \\ \leq Ct^{k+1} (1 + |Z^0| + |Z'^0|)^{2(n+k+m'+2)+m} \exp\left(\frac{1}{2}\nu u - \frac{2C''}{u}|Z - Z'|^2\right). \end{aligned}$$

*Proof.* By (2.160) and (2.167),

$$(2.172) \quad \frac{1}{r!} \frac{\partial^r}{\partial t^r} F_u(\mathcal{L}_2^t)|_{t=0} = F_{r,u}, \quad \frac{1}{r!} \frac{\partial^r}{\partial t^r} e^{-u\mathcal{L}_2^t}|_{t=0} = J_{r,u}.$$

Now by Theorem 2.17 and (2.160),  $J_{r,u}, F_{r,u}$  have the same estimates as  $\frac{\partial^r}{\partial t^r} e^{-u\mathcal{L}_2^t}$ ,  $\frac{\partial^r}{\partial t^r} F_u(\mathcal{L}_2^t)$  in (2.133), (2.158).

Again from (2.133), (2.158), (2.160), (2.167), and the Taylor expansion

$$(2.173) \quad G(t) - \sum_{r=0}^k \frac{1}{r!} \frac{\partial^r G}{\partial t^r}(0) t^r = \frac{1}{k!} \int_0^t (t-t_0)^k \frac{\partial^{k+1} G}{\partial t^{k+1}}(t_0) dt_0,$$

we get (2.171).  $\square$

**2.8. Evaluation of  $J_{r,u}$ .** For  $u > 0$ , we will write  $u\Delta_j$  for the rescaled simplex  $\{(u_1, \dots, u_j) \mid 0 \leq u_1 \leq u_2 \leq \dots \leq u_j \leq u\}$ .

Let  $e^{-u\mathcal{L}_2^0}(Z, Z')$  be the smooth kernel of  $e^{-u\mathcal{L}_2^0}$  with respect to  $dv_{T_{x_0}B}(Z')$ .

Recall that the  $\mathcal{O}_r$ 's have been defined in (2.102).

**Theorem 2.22.** *For  $r \geq 0$ , we have*

$$(2.174) \quad J_{r,u} = \sum_{\sum_{i=1}^j r_i = r, r_i \geq 1} (-1)^j \int_{u\Delta_j} e^{-(u-u_j)\mathcal{L}_2^0} \mathcal{O}_{r_j} e^{-(u_j-u_{j-1})\mathcal{L}_2^0} \dots \mathcal{O}_{r_1} e^{-u_1\mathcal{L}_2^0} du_1 \dots du_j,$$

where the product in the integrand is the convolution product. Moreover,

$$(2.175) \quad J_{r,u}(Z, Z') = (-1)^r J_{r,u}(-Z, -Z').$$

*Proof.* We introduce an even extra-variable  $\sigma$  such that  $\sigma^{r+1} = 0$ .

Set  $[\ ]^{[r]}$  the coefficient of  $\sigma^r$ ,  $\mathcal{L}_\sigma = \mathcal{L}_2^0 + \sum_{j=1}^r \mathcal{O}_j \sigma^j$ .

From (2.160), (2.172), we know

$$(2.176) \quad J_{r,u}(Z, Z') = \frac{1}{r!} \frac{\partial^r}{\partial t^r} e^{-u\mathcal{L}_2^t}(Z, Z')|_{t=0} = [e^{-u\mathcal{L}_\sigma}]^{[r]}(Z, Z').$$

Now from (2.176) and the Volterra expansion of  $e^{-u\mathcal{L}_\sigma}$  (cf. [1, §2.4]), we get (2.174).

We prove (2.175) by iteration.

From (2.102)

$$(2.177) \quad \mathcal{L}_2^0 = - \sum_{j=1}^{2n-n_0} (\nabla_{e_j})^2 - \pi^2 \langle ((P^{THU} \mathbf{J} P^{THU})^2 + 4P^{THU} \mathbf{J} P^{TY} \mathbf{J} P^{THU})_{x_0} \mathcal{R}, \mathcal{R} \rangle \\ + 2\pi \sqrt{-1} \nabla_{P^{THU} \mathbf{J} P^{THU} \mathcal{R}} - 2\omega_{d,x_0} - \tau_{x_0}.$$

Here the matrix  $((P^{THU} \mathbf{J} P^{THU})^2 + 4P^{THU} \mathbf{J} P^{TY} \mathbf{J} P^{THU})_{x_0}$  need not commute with  $P^{THU} \mathbf{J} P^{THU}$ .

Thus [3, (6.37), (6.38)] does not apply directly here, and we could not get a precise formula for  $e^{-u\mathcal{L}_2^0}$  as in [17, (4.106)].

By the uniqueness of the solution of heat equations and (2.177), we know

$$(2.178) \quad e^{-u\mathcal{L}_2^0}(Z, Z') = e^{-u\mathcal{L}_2^0}(-Z, -Z').$$

By (2.174),

$$(2.179) \quad J_{0,u}(Z, Z') = e^{-u\mathcal{L}_2^0}(Z, Z').$$



Thus we get (2.175) for  $r = 0$ .

If (2.175) holds for  $r \leq k$ , then by (2.174), (2.179),

$$(2.180) \quad J_{k+1,u} = - \sum_{j=1}^{k+1} \int_0^u e^{-(u-u_1)\mathcal{L}_2^0} \mathcal{O}_j J_{k+1-j,u_1} du_1.$$

By the iteration, Theorem 2.11 and (2.179), and note that  $\nabla_{e_i}$  in  $\mathcal{O}_j$  will change the parity of the polynomials we obtained, we get (2.175) for  $r = k + 1$ .  $\square$

**2.9. Proof of Theorem 0.2.** By (2.157) and (2.171), for any  $u > 0$  fixed, there exists  $C_u > 0$  such that for  $t = \frac{1}{\sqrt{p}}$ ,  $Z, Z' \in T_{x_0}B$ ,  $x_0 \in P$ ,  $\alpha, \alpha' \in \mathbb{Z}^{2n-n_0}$ ,  $|\alpha| + |\alpha'| \leq m$ , we have

$$(2.181) \quad (1 + |Z^\perp| + |Z'^\perp|)^{m''} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left( P_{0,t} - \sum_{r=0}^k t^r (J_{r,u} - F_{r,u}) \right) (Z, Z') \right|_{\mathcal{C}^{m'}(X_G)} \\ \leq C_u t^{k+1} (1 + |Z^0| + |Z'^0|)^{2(n+k+m'+2)+m} \exp(-\sqrt{C''\nu}|Z - Z'|).$$

Set

$$(2.182) \quad P^{(r)} = J_{r,u} - F_{r,u}.$$

Then  $P^{(r)}$  does not depend on  $u > 0$  by (2.181), as  $P_{0,t}$  does not depend on  $u$ .

Moreover, by taking the limit of (2.158) as  $t \rightarrow 0$ ,

$$(2.183) \quad (1 + |Z^\perp| + |Z'^\perp|)^{m''} \left| F_{r,u}(Z, Z') \right|_{\mathcal{C}^{m'}(X_G)} \\ \leq C(1 + |Z^0| + |Z'^0|)^{2n+2r+2m'+2} \exp(-\frac{1}{8}\nu u - \sqrt{C''\nu}|Z - Z'|).$$

Thus

$$(2.184) \quad J_{r,u}(Z, Z') = P^{(r)}(Z, Z') + F_{r,u}(Z, Z') = P^{(r)}(Z, Z') + \mathcal{O}(e^{-\frac{1}{8}\nu u}),$$

uniformly on any compact set of  $T_{x_0}B \times T_{x_0}B$ .

Especially, from (2.175), (2.184), we get

$$(2.185) \quad P^{(r)}(Z, Z') = (-1)^r P^{(r)}(-Z, -Z').$$

By (2.101), for  $Z, Z' \in T_{x_0}B$ ,

$$(2.186) \quad P_{x_0,p}(Z, Z') = p^{n-\frac{n_0}{2}} \kappa^{-\frac{1}{2}}(Z) P_{0,t}(Z/t, Z'/t) \kappa^{-\frac{1}{2}}(Z').$$

We note in passing that, as a consequence of (2.181) and (2.186), we obtain the following estimate.

**Theorem 2.23.** *For any  $k, m, m', m'' \in \mathbb{N}$ , there exists  $C > 0$  such that for  $Z, Z' \in T_{x_0}B$ ,  $|Z|, |Z'| \leq \varepsilon$ ,  $x_0 \in X_G$ ,*

$$(2.187) \quad \sup_{|\alpha|+|\alpha'| \leq m} (1 + \sqrt{p}|Z^\perp| + \sqrt{p}|Z'^\perp|)^{m''} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \right. \\ \left. \left( p^{-n+\frac{n_0}{2}} P_{x_0,p}(Z, Z') - \sum_{r=0}^k P^{(r)}(\sqrt{p}Z, \sqrt{p}Z') \kappa^{-\frac{1}{2}}(Z) \kappa^{-\frac{1}{2}}(Z') p^{-r/2} \right) \right|_{\mathcal{C}^{m'}(X_G)} \\ \leq C p^{-(k+1-m)/2} (1 + \sqrt{p}|Z^0| + \sqrt{p}|Z'^0|)^{2(n+k+m'+2)+m} \exp(-\sqrt{C''}\nu\sqrt{p}|Z - Z'|).$$

From (2.84), (2.85), (2.109) and (2.187), we get Theorem 0.2 without knowing the properties (0.12), (0.13) for  $P^{(r)}$ .

To prove the uniformity part of Theorem 0.2, we notice that in the proof of Theorem 2.17, we only use the derivatives of the coefficients of  $\mathcal{L}_2^t$  with order  $\leq 2n+m+m'+r+2$ . Thus the constants in Theorems 2.17 and 2.20, (resp. Theorem 2.21) are uniformly bounded, if with respect to a fixed metric  $g_0^{TX}$ , the  $\mathcal{C}^{2n+m+m'+r+2}$  (resp.  $\mathcal{C}^{2n+m+m'+k+3}$ ) – norms on  $X$  of the data  $(g^{TX}, h^L, \nabla^L, h^E, \nabla^E, J)$  are bounded, and  $g^{TX}$  is bounded below.

Moreover, taking derivatives with respect to the parameters we obtain a similar equation as (2.155), where  $x_0 \in X_G$  plays now a role of a parameter. Thus the  $\mathcal{C}^{m'}$ –norm in (2.187) can also include the parameters if the  $\mathcal{C}^{m'}$ –norms (with respect to the parameter  $x_0 \in X_G$ ) of the derivatives of above data with order  $\leq 2n+k+m+3$  are bounded.

Thus we can take  $C_{k,l}$  in (0.10) independent of  $g^{TX}$  under our condition.

This achieves the proof of Theorem 0.2 except (0.12) and (0.13) which will be proved in Theorem 3.2 under the condition in Theorem 0.2.

3. EVALUATION OF  $P^{(r)}$ 

In this Section, inspired by the method in [26, §1.4, 1.5], we develop a direct and effective method to compute  $P^{(r)}$ . In particular, we get (0.12) and (0.13) under the condition in Theorem 0.2.

This section is organized as follows. In Section 3.1, we study the spectrum of the limited operator  $\mathcal{L}_2^0$ . In Section 3.2, we get a direct method to evaluate  $P^{(r)}$  in (0.12), especially, we prove (0.12) and (0.13). In Section 3.3, we compute explicitly  $\mathcal{O}_1$  in (2.103), and get a general formula for  $P^{(2)}$  by using the operators  $\mathcal{O}_1, \mathcal{O}_2$ . In Section 3.4, we compute explicitly an interesting example: the line bundle  $\mathcal{O}(2)$  on  $(\mathbb{C}P^1, 2\omega_{FS})$ . We verify that Theorem 0.2 coincides with our computation here if 0 is a regular value of the moment map  $\mu$ , but it does not hold if 0 is a singular value.

We use the notations in Section 2.6, and we suppose that (3.2) is verified.

**3.1. Spectrum of  $\mathcal{L}_2^0$ .** Recall that  $T^H P$  is the orthogonal complement of  $TY$  in  $(TP, g^{TP})$ . Note that by (2.5) and (2.15), we have the following orthogonal splitting of vector bundles on  $P = \mu^{-1}(0)$ ,

$$(3.1) \quad TX = T^H P \oplus TY \oplus \mathbf{J}TY, \quad TP = T^H P \oplus TY.$$

In the rest of this Section, we suppose that on  $P$

$$(3.2) \quad \mathbf{J}T^H P = T^H P, \quad \mathbf{J}^2 TY = TY.$$

(2.7) and (3.2) imply that  $-\mathbf{J}\mathbf{J}$  preserves  $TY$  and  $\mathbf{J}TY$ .

As  $g^{TX}$  is  $J$ -invariant, we get

$$(3.3) \quad \mathbf{J}TY = JTY, \quad JT^H P = T^H P.$$

Thus  $(\mathbf{J}TY)_B|_{X_G}$  is the orthogonal complement of  $TX_G$  in  $TB$ , and  $\mathbf{J}$  induces naturally  $\mathbf{J}_G \in \text{End}(TX_G)$ . We will identify  $(\mathbf{J}TY)_B|_{X_G}$  to the normal bundle of  $X_G$  in  $B$ .

For  $U, V \in T_{x_0}B$ ,  $x_0 \in X_G$ , we have

$$(3.4) \quad \omega(U^H, V^H) = \omega_G(P^{TX_G}U, P^{TX_G}V).$$

From the above discussion, for  $x_0 \in X_G$ , we can choose  $\{w_j^0\}_{j=1}^{n-n_0}, \{e_j^\perp\}_{j=1}^{n_0}$  orthonormal basis of  $T_{x_0}^{(1,0)}X_G$ ,  $(\mathbf{J}TY)_{B,x_0} \subset TB$  such that

$$(3.5) \quad \begin{aligned} \mathbf{J}|_{T_{x_0}^{(1,0)}X_G} &= \frac{\sqrt{-1}}{2\pi} \text{diag}(a_1, \dots, a_{n-n_0}) \in \text{End}(T_{x_0}^{(1,0)}X_G), \\ \mathbf{J}^2|_{(\mathbf{J}TY)_B} &= \frac{-1}{4\pi^2} \text{diag}(a_1^{\perp,2}, \dots, a_{n_0}^{\perp,2}) \in \text{End}((\mathbf{J}TY)_{B,x_0}), \end{aligned}$$

with  $a_j, a_j^\perp > 0$ , and let  $\{w_j^{0,j}\}_{j=1}^{n-n_0}, \{e_j^{\perp,j}\}_{j=1}^{n_0}$  be their dual basis, then

$$e_{2j-1}^0 = \frac{1}{\sqrt{2}}(w_j^0 + \overline{w}_j^0) \quad \text{and} \quad e_{2j}^0 = \frac{\sqrt{-1}}{\sqrt{2}}(w_j^0 - \overline{w}_j^0),$$

$j = 1, \dots, n - n_0$ , forms an orthonormal basis of  $T_{x_0}X_G$ .

From now on, we use the coordinate in Section 2.6 induced by the above basis.

Denote by  $Z^0 = (Z_1^0, \dots, Z_{2n-2n_0}^0)$ ,  $Z^\perp = (Z_1^\perp, \dots, Z_{n_0}^\perp)$ , then  $Z = (Z^0, Z^\perp)$ .

In what follows we will use the complex coordinates  $z^0 = (z_1^0, \dots, z_{n-n_0}^0)$ , thus  $Z^0 = z^0 + \bar{z}^0$ , and  $w_i^0 = \sqrt{2} \frac{\partial}{\partial z_i^0}$ ,  $\bar{w}_i^0 = \sqrt{2} \frac{\partial}{\partial \bar{z}_i^0}$ , and

$$(3.6) \quad e_{2i-1}^0 = \frac{\partial}{\partial z_i^0} + \frac{\partial}{\partial \bar{z}_i^0}, \quad e_{2i}^0 = \sqrt{-1} \left( \frac{\partial}{\partial z_i^0} - \frac{\partial}{\partial \bar{z}_i^0} \right).$$

We will also identify  $z^0$  to  $\sum_i z_i^0 \frac{\partial}{\partial z_i^0}$  and  $\bar{z}^0$  to  $\sum_i \bar{z}_i^0 \frac{\partial}{\partial \bar{z}_i^0}$  when we consider  $z^0$  and  $\bar{z}^0$  as vector fields. Remark that

$$(3.7) \quad \left| \frac{\partial}{\partial z_i^0} \right|^2 = \left| \frac{\partial}{\partial \bar{z}_i^0} \right|^2 = \frac{1}{2}, \quad \text{so that } |z^0|^2 = |\bar{z}^0|^2 = \frac{1}{2} |Z^0|^2.$$

It is very useful to rewrite  $\mathcal{L}_2^0$  by using the creation and annihilation operators. Set

$$(3.8) \quad \begin{aligned} b_i &= -2 \frac{\partial}{\partial z_i^0} + \frac{1}{2} a_i \bar{z}_i^0, & b_i^+ &= 2 \frac{\partial}{\partial \bar{z}_i^0} + \frac{1}{2} a_i z_i^0, & b &= (b_1, \dots, b_{n-n_0}); \\ b_j^\perp &= -\frac{\partial}{\partial Z_j^\perp} + a_j^\perp Z_j^\perp, & b_j^{\perp+} &= \frac{\partial}{\partial Z_j^\perp} + a_j^\perp Z_j^\perp, & b^\perp &= (b_1^\perp, \dots, b_{n_0}^\perp). \end{aligned}$$

Then for any polynomial  $g(Z^0, Z^\perp)$  on  $Z^0$  and  $Z^\perp$ ,

$$(3.9) \quad \begin{aligned} [b_i, b_j^+] &= b_i b_j^+ - b_j^+ b_i = -2a_i \delta_{ij}, & [b_i, b_j] &= [b_i^+, b_j^+] = 0, \\ [g, b_j] &= 2 \frac{\partial}{\partial \bar{z}_j^0} g, & [g, b_j^+] &= -2 \frac{\partial}{\partial z_j^0} g, \\ [b_i^\perp, b_j^{\perp+}] &= -2a_i^\perp \delta_{ij}, & [b_j^\perp, b_k^\perp] &= [b_j^{\perp+}, b_k^{\perp+}] = 0, \\ [g, b_j^\perp] &= -[g, b_j^{\perp+}] = \frac{\partial}{\partial Z_j^\perp} g. \end{aligned}$$

Set

$$(3.10) \quad \mathcal{L} = \sum_{j=1}^{n-n_0} b_j b_j^+, \quad \mathcal{L}^\perp = \sum_{j=1}^{n_0} b_j^\perp b_j^{\perp+}, \quad \nabla_{0,\cdot} = \nabla \cdot + \frac{1}{2} R_{x_0}^{LB}(\mathcal{R}, \cdot).$$

From (1.18) and (3.4), for  $U, V \in T_{x_0}B$ , we get

$$(3.11) \quad R_{x_0}^{LB}(U, V) = -2\pi\sqrt{-1} \langle \mathbf{J} P^{TX_G} U, P^{TX_G} V \rangle.$$

By (2.48), (3.5), (3.8), (3.10) and (3.11), we have

$$(3.12) \quad \begin{aligned} b_i &= -2\nabla_{0, \frac{\partial}{\partial z_i^0}}, & b_i^+ &= 2\nabla_{0, \frac{\partial}{\partial \bar{z}_i^0}}, & \nabla_{0, e_j^\perp} &= \nabla_{e_j^\perp}, \\ \tau_{x_0} &= \sum_j a_j + \sum_j a_j^\perp. \end{aligned}$$

From (2.102), (3.10) and (3.12), we get

$$(3.13) \quad \begin{aligned} \mathcal{L}_2^0 &= - \sum_{j=1}^{2n-2n_0} (\nabla_{0, e_j^0})^2 - \sum_{j=1}^{n_0} \left( (\nabla_{e_j^\perp})^2 - |a_j^\perp Z_j^\perp|^2 \right) - 2\omega_{d, x_0} - \tau_{x_0} \\ &= \mathcal{L} + \mathcal{L}^\perp - 2\omega_{d, x_0}. \end{aligned}$$

By [38, §8.6], [26, Theorem 1.15], we know

**Theorem 3.1.** *The spectrum of the restriction of  $\mathcal{L}$  on  $L^2(\mathbb{R}^{2n-2n_0})$  is given by*

$$(3.14) \quad \text{Spec}(\mathcal{L}|_{L^2(\mathbb{R}^{2n-2n_0})}) = \left\{ 2 \sum_{i=1}^{n-n_0} \alpha_i^0 a_i : \alpha^0 = (\alpha_1^0, \dots, \alpha_{n-n_0}^0) \in \mathbb{N}^{n-n_0} \right\},$$

and an orthogonal basis of the eigenspace of  $2 \sum_{i=1}^{n-n_0} \alpha_i^0 a_i$  is given by

$$(3.15) \quad b^{\alpha^0} \left( (z^0)^\beta \exp \left( -\frac{1}{4} \sum_i a_i |z_i^0|^2 \right) \right), \quad \text{with } \beta \in \mathbb{N}^{n-n_0}.$$

The spectrum of the restriction of  $\mathcal{L}^\perp$  on  $L^2(\mathbb{R}^{n_0})$  is given by

$$(3.16) \quad \text{Spec}(\mathcal{L}^\perp|_{L^2(\mathbb{R}^{n_0})}) = \left\{ 2 \sum_{i=1}^{n_0} \alpha_i^\perp a_i^\perp : \alpha^\perp = (\alpha_1^\perp, \dots, \alpha_{n_0}^\perp) \in \mathbb{N}^{n_0} \right\},$$

and the eigenspace of  $2 \sum_{i=1}^{n_0} \alpha_i^\perp a_i^\perp$  is one dimensional and an orthonormal basis is given by

$$(3.17) \quad \left( \prod_{i=1}^{n_0} \sqrt{\frac{\pi}{a_i^\perp}} (2a_i^\perp)^{\alpha_i^\perp} (\alpha_i^\perp!) \right)^{-1/2} (b^\perp)^{\alpha^\perp} \exp \left( -\frac{1}{2} \sum_i a_i^\perp |Z_i^\perp|^2 \right).$$

Especially, the orthonormal basis of  $\text{Ker}(\mathcal{L}|_{L^2(\mathbb{R}^{2n-2n_0})}); \text{Ker}(\mathcal{L}^\perp|_{L^2(\mathbb{R}^{n_0})})$  are

$$(3.18) \quad \left( \frac{a^\beta}{2^{|\beta|} \beta!} \prod_{i=1}^{n-n_0} \frac{a_i}{2\pi} \right)^{\frac{1}{2}} \left( (z^0)^\beta \exp \left( -\frac{1}{4} \sum_{j=1}^{n-n_0} a_j |z_j^0|^2 \right) \right), \quad \beta \in \mathbb{N}^{n-n_0};$$

$$G^\perp(Z^\perp) = \left( \prod_{i=1}^{n_0} \frac{a_i^\perp}{\pi} \right)^{\frac{1}{4}} \exp \left( -\frac{1}{2} \sum_{i=1}^{n_0} a_i^\perp |Z_i^\perp|^2 \right).$$

Let  $P_{\mathcal{L}}(Z^0, Z'^0)$ ,  $P_{\mathcal{L}^\perp}(Z^\perp, Z'^\perp)$  (resp.  $P(Z, Z')$ ) be the kernels of the orthogonal projections  $P_{\mathcal{L}}$ ,  $P_{\mathcal{L}^\perp}$  (resp.  $P$ ) from  $L^2(\mathbb{R}^{2n-2n_0})$  onto  $\text{Ker}(\mathcal{L})$ ,  $L^2(\mathbb{R}^{n_0})$  onto  $\text{Ker}(\mathcal{L}^\perp)$  (resp.  $L^2(\mathbb{R}^{2n-n_0})$  onto  $\text{Ker}(\mathcal{L} + \mathcal{L}^\perp)$ ).

From (3.18), we get

$$(3.19) \quad P_{\mathcal{L}}(Z^0, Z'^0) = \left( \prod_{i=1}^{n-n_0} \frac{a_i}{2\pi} \right) \exp \left( -\frac{1}{4} \sum_{i=1}^{n-n_0} a_i (|z_i^0|^2 + |z_i'^0|^2 - 2z_i^0 \bar{z}_i'^0) \right),$$

$$P_{\mathcal{L}^\perp}(Z^\perp, Z'^\perp) = \left( \prod_{i=1}^{n_0} \sqrt{\frac{a_i^\perp}{\pi}} \right) \exp \left( -\frac{1}{2} \sum_{i=1}^{n_0} a_i^\perp (|Z_i^\perp|^2 + |Z_i'^\perp|^2) \right),$$

$$P(Z, Z') = P_{\mathcal{L}}(Z^0, Z'^0) P_{\mathcal{L}^\perp}(Z^\perp, Z'^\perp).$$

Let  $P^N$  be the orthogonal projection from  $L^2(\mathbb{R}^{2n-n_0}, (\Lambda(T^{*(0,1)}X) \otimes E)_{x_0})$  onto  $N = \text{Ker}(\mathcal{L}_2^0)$ . Let  $P^N(Z, Z')$  be the associated kernel.

Recall that the projection  $I_{\mathbb{C} \otimes E_B}$  from  $(\Lambda(T^{*(0,1)}X) \otimes E)_B$  onto  $\mathbb{C} \otimes E_B$  is defined in Introduction.

By (2.7), (2.9), (2.48) and (3.5),

$$(3.20) \quad -\omega_{d,x_0} \geq \nu_0 \quad \text{on } \Lambda^{>0}(T^{*(0,1)}X),$$

thus

$$(3.21) \quad P^N(Z, Z') = P(Z, Z') I_{\mathbb{C} \otimes E_B}.$$

If  $\mathbf{J} = J$  on  $P$ , then by (3.19) and (3.21),

$$(3.22) \quad \begin{aligned} P^N(Z, Z') &= \exp \left( -\frac{\pi}{2} \sum_{i=1}^{n-n_0} (|z_i^0|^2 + |z_i'^0|^2 - 2z_i^0 \bar{z}_i'^0) \right) \\ &\quad \times 2^{\frac{n_0}{2}} \exp \left( -\pi (|Z^\perp|^2 + |Z'^\perp|^2) \right) I_{\mathbb{C} \otimes E_B}, \\ P^N((0, Z^\perp), (0, Z'^\perp)) &= 2^{\frac{n_0}{2}} \exp \left( -2\pi |Z^\perp|^2 \right) I_{\mathbb{C} \otimes E_B}. \end{aligned}$$

**3.2. Evaluation of  $P^{(r)}$ : a proof of (0.12) and (0.13).** Recall that  $\delta$  is the counter-clockwise oriented circle in  $\mathbb{C}$  of center 0 and radius  $\nu/4$ .

By (2.121),

$$(3.23) \quad P_{0,t} = \frac{1}{2\pi i} \int_{\delta} (\lambda - \mathcal{L}_2^t)^{-1} d\lambda.$$

Let  $f(\lambda, t)$  be a formal power series with values in  $\text{End}(L^2(\mathbb{R}^{2n-n_0}, (\Lambda(T^{*(0,1)}X) \otimes E)_{B,x_0}))$

$$(3.24) \quad f(\lambda, t) = \sum_{r=0}^{\infty} t^r f_r(\lambda), \quad f_r(\lambda) \in \text{End}(L^2(\mathbb{R}^{2n-n_0}, (\Lambda(T^{*(0,1)}X) \otimes E)_{B,x_0})).$$

By (2.103), consider the equation of formal power series for  $\lambda \in \delta$ ,

$$(3.25) \quad (\lambda - \mathcal{L}_2^0 - \sum_{r=1}^{\infty} t^r \mathcal{O}_r) f(\lambda, t) = \text{Id}_{L^2(\mathbb{R}^{2n-n_0}, (\Lambda(T^{*(0,1)}X) \otimes E)_{B,x_0})}.$$

Let  $N^\perp$  be the orthogonal space of  $N$  in  $L^2(\mathbb{R}^{2n-n_0}, (\Lambda(T^{*(0,1)}X) \otimes E)_{B,x_0})$ , and  $P^{N^\perp}$  be the orthogonal projection from  $L^2(\mathbb{R}^{2n-n_0}, (\Lambda(T^{*(0,1)}X) \otimes E)_{B,x_0})$  onto  $N^\perp$ .

We decompose  $f(\lambda, t)$  according the splitting  $L^2(\mathbb{R}^{2n-n_0}, (\Lambda(T^{*(0,1)}X) \otimes E)_{B,x_0}) = N \oplus N^\perp$ ,

$$(3.26) \quad g_r(\lambda) = P^N f_r(\lambda), \quad f_r^\perp(\lambda) = P^{N^\perp} f_r(\lambda).$$

Using Theorem 3.1, (3.13), (3.20), (3.26) and identifying the powers of  $t$  in (3.25), we find that

$$(3.27) \quad \begin{aligned} g_0(\lambda) &= \frac{1}{\lambda} P^N, \quad f_0^\perp(\lambda) = (\lambda - \mathcal{L}_2^0)^{-1} P^{N^\perp}, \\ f_r^\perp(\lambda) &= (\lambda - \mathcal{L}_2^0)^{-1} \sum_{j=1}^r P^{N^\perp} \mathcal{O}_j f_{r-j}(\lambda), \\ g_r(\lambda) &= \frac{1}{\lambda} \sum_{j=1}^r P^N \mathcal{O}_j f_{r-j}(\lambda). \end{aligned}$$

Recall that  $P^{(r)}$  ( $r \in \mathbb{N}$ ) is defined in (2.182) and (2.187).

**Theorem 3.2.** *There exist  $J_r(Z, Z')$  polynomials in  $Z, Z'$  with the same parity as  $r$ , whose coefficients are polynomials in  $A, R^{TB}, R^{\text{Cliff}_B}, R^{E_B}, \mu^E, \mu^{\text{Cliff}}$  (resp.  $r^X, \text{Tr}[R^{T^{(1,0)}X}]$ ,  $R^E$ ; resp.  $h, R^L, R^{L_B}$ ; resp.  $\mu$ ) and their derivatives at  $x_0$  up to order  $r-1$  (resp.  $r-2$ ;*

resp.  $r$ ; resp.  $r + 1$ ), and in the inverses of the linear combination of the eigenvalues of  $\mathbf{J}$  at  $x_0$ , such that

$$(3.28) \quad P^{(r)}(Z, Z') = J_r(Z, Z')P(Z, Z').$$

Moreover,

$$(3.29) \quad P^{(0)}(Z, Z') = P^N(Z, Z') = P(Z, Z')I_{\mathbb{C} \otimes E_B}.$$

*Proof.* By (3.23), for  $M > 0$ , by combining Theorems 2.13-2.16 and the argument as in [26, §1.3], we get another proof of the existence of the asymptotic expansion of  $P_{0,t}(Z, Z')$  for  $|Z|, |Z'| \leq M$  when  $t \rightarrow 0$ .

By (2.84), (2.85) and (2.186), this gives another proof of Theorems 0.2, 2.23 for  $|Z|, |Z'| \leq M/\sqrt{p}$ . Moreover, by (2.150), (2.160) and (3.26),

$$(3.30) \quad P^{(r)} = \frac{1}{2\pi i} \int_{\delta} g_r(\lambda) d\lambda + \frac{1}{2\pi i} \int_{\delta} f_r^{\perp}(\lambda) d\lambda.$$

From (3.27), (3.30), we get (3.29).

Generally, from Theorems 2.11, 3.1, (3.9), (3.27), (3.30) and the residue formula, we conclude Theorem 3.2.  $\square$

*Proof of (0.12) and (0.13).* As  $\mathbf{J} = J$  on  $\mu^{-1}(0)$ , the condition (3.2) is verified.

From Theorem 3.2, (3.22), we get (0.12) and (0.13).  $\square$

From Theorem 3.1, (3.27), (3.30), and the residue formula, we can get  $P^{(r)}$  by using the operators  $(\mathcal{L}_2^0)^{-1}$ ,  $P^N$ ,  $P^{N^{\perp}}$ ,  $\mathcal{O}_k$  ( $k \leq r$ ).

This gives us a direct method to compute  $P^{(r)}$  in view of Theorem 3.1. In particular,

$$(3.31) \quad P^{(1)} = -P^N \mathcal{O}_1 P^{N^{\perp}} (\mathcal{L}_2^0)^{-1} P^{N^{\perp}} - P^{N^{\perp}} (\mathcal{L}_2^0)^{-1} P^{N^{\perp}} \mathcal{O}_1 P^N,$$

and

$$(3.32) \quad \begin{aligned} P^{(2)} &= \frac{1}{2\pi i} \int_{\delta} \left[ (\lambda - \mathcal{L}_2^0)^{-1} P^{N^{\perp}} (\mathcal{O}_1 f_1 + \mathcal{O}_2 f_0)(\lambda) + \frac{1}{\lambda} P^N (\mathcal{O}_1 f_1 + \mathcal{O}_2 f_0)(\lambda) \right] d\lambda \\ &= \frac{1}{2\pi i} \int_{\delta} \left\{ (\lambda - \mathcal{L}_2^0)^{-1} P^{N^{\perp}} \left[ \mathcal{O}_1 \left( (\lambda - \mathcal{L}_2^0)^{-1} P^{N^{\perp}} \mathcal{O}_1 + \frac{1}{\lambda} P^N \mathcal{O}_1 \right) + \mathcal{O}_2 \right] \right. \\ &\quad \left. + \frac{1}{\lambda} P^N \left[ \mathcal{O}_1 \left( (\lambda - \mathcal{L}_2^0)^{-1} P^{N^{\perp}} \mathcal{O}_1 + \frac{1}{\lambda} P^N \mathcal{O}_1 \right) + \mathcal{O}_2 \right] \right\} (\lambda - \mathcal{L}_2^0)^{-1} d\lambda \\ &= (\mathcal{L}_2^0)^{-1} P^{N^{\perp}} \mathcal{O}_1 (\mathcal{L}_2^0)^{-1} P^{N^{\perp}} \mathcal{O}_1 P^N - P^{N^{\perp}} (\mathcal{L}_2^0)^{-2} \mathcal{O}_1 P^N \mathcal{O}_1 P^N \\ &\quad + (\mathcal{L}_2^0)^{-1} P^{N^{\perp}} \mathcal{O}_1 P^N \mathcal{O}_1 (\mathcal{L}_2^0)^{-1} P^{N^{\perp}} - (\mathcal{L}_2^0)^{-1} P^{N^{\perp}} \mathcal{O}_2 P^N \\ &\quad + P^N \mathcal{O}_1 (\mathcal{L}_2^0)^{-1} P^{N^{\perp}} \mathcal{O}_1 (\mathcal{L}_2^0)^{-1} P^{N^{\perp}} - P^N \mathcal{O}_1 (\mathcal{L}_2^0)^{-2} P^{N^{\perp}} \mathcal{O}_1 P^N \\ &\quad - P^N \mathcal{O}_1 P^N \mathcal{O}_1 (\mathcal{L}_2^0)^{-2} P^{N^{\perp}} - P^N \mathcal{O}_2 (\mathcal{L}_2^0)^{-1} P^{N^{\perp}}. \end{aligned}$$

In the next Subsection we will prove  $P^N \mathcal{O}_1 P^N = 0$ , thus the second and seventh terms in (3.32) are zero.

**3.3. A formula for  $\mathcal{O}_1$ .** We will use the notation in Section 1. All tensors in this Subsection will be evaluated at the base point  $x_0 \in X_G$ .

For  $\psi$  a tensor on  $X$ , we denote by  $\nabla^X \psi$  its covariant derivative induced by  $\nabla^{TX}$ .

If  $\psi_1$  is a  $G$ -equivariant tensor, then we can consider it as a tensor on  $B = U/G$  with the covariant derivative  $\nabla^B \psi_1$ , we will denote by

$$(\nabla^B \nabla^B \psi_1)_{(c_j e_j, c'_k e_k)} := c_j c'_k (\nabla_{e_j}^B \nabla_{e_k}^B \psi_1)_{x_0},$$

etc.

We denote by  $\{e_a\}$  an orthonormal basis of  $(TX, g^{TX})$ .

To simplify the notation, we often denote by  $U$  the lift  $U^H$  of  $U \in TB$ .

Recall that  $\tilde{\mu} \in TY$  is defined by (1.14) and the moment map  $\mu$  (2.14), and that  $A$  is the second fundamental form of  $X_G$  defined by (0.10).

**Lemma 3.3.** *The following identities hold,*

$$\begin{aligned} (3.33) \quad & (\nabla_{\mathcal{R}}^{TY} \tilde{\mu})_{x_0} = -\mathbf{J}\mathcal{R}^\perp, \\ & (\nabla_{\cdot}^{TY} \nabla_{\cdot}^{TY} \tilde{\mu})_{(\mathcal{R}, \mathcal{R})} := (\nabla_{e_j^H}^{TY} \nabla_{e_i^H}^{TY} \tilde{\mu})_{x_0} Z_j Z_i \\ & = -P^{TY} ((\nabla_{\mathcal{R}^0}^X \mathbf{J})(\mathcal{R}^0 + 2\mathcal{R}^\perp) + (\nabla_{\mathcal{R}^\perp}^X \mathbf{J})\mathcal{R}^\perp) \\ & \quad - \mathbf{J}A(\mathcal{R}^0)\mathcal{R}^0 - \frac{1}{2}T(\mathcal{R}^0, \mathbf{J}\mathcal{R}^0) + T(\mathcal{R}^\perp, \mathbf{J}\mathcal{R}^\perp). \end{aligned}$$

*Proof.* Recall that  $P^{TY}, P^{T^H X}$  are the orthogonal projections from  $TX$  onto  $TY, T^H X$  defined in Section 1.1. Note that on  $P$ , by (3.3),

$$(3.34) \quad \mathbf{J}e_i^{\perp, H} \in TY, \quad \mathbf{J}e_i^{0, H} = (\mathbf{J}_G e_i^0)^H \in T^H P.$$

By (1.14) and (2.15), for  $K \in \mathfrak{g}$ ,

$$(3.35) \quad -\langle \mathbf{J}e_i^H, K^X \rangle = \nabla_{e_i^H} \mu(K) = \left\langle \nabla_{e_i^H}^{TY} \tilde{\mu}, K^X \right\rangle + \left\langle \tilde{\mu}, \nabla_{e_i^H}^{TY} K^X \right\rangle.$$

From (1.4), (1.5), (1.6) and (3.35),

$$(3.36) \quad \nabla_{e_i^H}^{TY} \tilde{\mu} = -P^{TY} \mathbf{J}e_i^H - \frac{1}{2} \dot{g}_{e_i^H}^{TY} \tilde{\mu} = -P^{TY} \mathbf{J}e_i^H - T(e_i^H, \tilde{\mu}).$$

From (3.36) and the fact that  $\tilde{\mu} = 0$  on  $P$ , one gets the first equation in (3.33).

Now for  $W$  (resp.  $Y$ ) a smooth section of  $TX$  (resp.  $TY$ ), by (1.8),

$$\begin{aligned} (3.37) \quad \left\langle \nabla_{e_j^H}^{TY} P^{TY} W, Y \right\rangle &= e_j^H \langle W, Y \rangle - \left\langle P^{TY} W, \nabla_{e_j^H}^{TY} Y \right\rangle \\ &= \left\langle \nabla_{e_j^H}^{TX} W, Y \right\rangle + \frac{1}{2} \left\langle T(e_j^H, P^{T^H X} W), Y \right\rangle. \end{aligned}$$

By (3.37),

$$(3.38) \quad \nabla_{e_j^H}^{TY} P^{TY} W = P^{TY} \nabla_{e_j^H}^{TX} W + \frac{1}{2} T(e_j^H, P^{T^H X} W).$$



By (3.36) and (3.38),

$$(3.39) \quad \nabla_{e_j^H}^{TY} \nabla_{e_i^H}^{TY} \tilde{\mu} = -P^{TY} (\nabla_{e_j^H}^X \mathbf{J}) e_i^H - P^{TY} \mathbf{J} \nabla_{e_j^H}^{TX} e_i^H \\ - \frac{1}{2} T(e_j^H, P^{TX} \mathbf{J} e_i^H) - \frac{1}{2} (\nabla_{e_j^H}^{TY} \dot{g}_{e_i^H}^{TY}) \tilde{\mu} - \frac{1}{2} \dot{g}_{e_i^H}^{TY} (\nabla_{e_j^H}^{TY} \tilde{\mu}).$$

By (1.3) and (1.7), for  $U_1, U_2$  sections of  $TB$  on  $B$ ,

$$(3.40) \quad \nabla_{U_2^H}^{TX} U_1^H = (\nabla_{U_2}^{TB} U_1)^H - \frac{1}{2} T(U_2^H, U_1^H).$$

By the definition of our basis  $\{e_i^0, e_j^\perp\}$  in Section 2.6,

$$(3.41) \quad (\nabla_{e_i^0}^{TB} e_j^0)_{x_0} = A(e_i^0) e_j^0, \quad (\nabla_{e_i^0}^{TB} e_j^\perp)_{x_0} = (\nabla_{e_j^\perp}^{TB} e_i^0)_{x_0} = A(e_i^0) e_j^\perp, \quad (\nabla_{e_j^\perp}^{TB} e_i^\perp)_{x_0} = 0.$$

Thus by (1.6), (3.2), (3.36), (3.39), (3.40), (3.41) and the facts that  $A$  exchanges  $N_G$  and  $TX_G$  on  $X_G$ , and that  $\tilde{\mu} = 0$  on  $P$ , we get

$$(3.42) \quad (\nabla_{\mathcal{R}}^{TY} \nabla_{\mathcal{R}}^{TY} \tilde{\mu})_{(\mathcal{R}, \mathcal{R})} = -P^{TY} (\nabla_{\mathcal{R}}^X \mathbf{J}) \mathcal{R} - \mathbf{J} A(\mathcal{R}^0) \mathcal{R}^0 - \frac{1}{2} T(\mathcal{R}, \mathbf{J} \mathcal{R}^0) + T(\mathcal{R}, \mathbf{J} \mathcal{R}^\perp).$$

We use the closeness of  $\omega$  to complete the proof of (3.33).

From (0.2), for  $U, V, W \in TX$ ,

$$(3.43) \quad \langle (\nabla_U^X \mathbf{J}) V, W \rangle = (\nabla_U^X \omega)(V, W),$$

thus

$$(3.44) \quad \langle (\nabla_U^X \mathbf{J}) V, W \rangle + \langle (\nabla_V^X \mathbf{J}) W, U \rangle + \langle (\nabla_W^X \mathbf{J}) U, V \rangle = d\omega(U, V, W) = 0.$$

By (1.3), (1.7), (3.44) and (3.34), for  $Y$  a smooth section of  $TY$ ,

$$\langle \mathbf{J} \nabla_Y^{TX} e_j^0, e_i^\perp \rangle = -\langle \nabla_Y^{TX} e_j^0, \mathbf{J} e_i^\perp \rangle = -\langle T(e_j^0, \mathbf{J} e_i^\perp), Y \rangle$$

and

$$(3.45) \quad \langle T(e_i^\perp, \mathbf{J} e_j^0), Y \rangle = -2 \langle \nabla_Y^{TX} (\mathbf{J} e_j^0), e_i^\perp \rangle = -2 \langle (\nabla_Y^X \mathbf{J}) e_j^0, e_i^\perp \rangle + 2 \langle T(e_j^0, \mathbf{J} e_i^\perp), Y \rangle \\ = 2 \langle (\nabla_{e_j^0}^X \mathbf{J}) e_i^\perp, Y \rangle - 2 \langle (\nabla_{e_i^\perp}^X \mathbf{J}) e_j^0, Y \rangle + 2 \langle T(e_j^0, \mathbf{J} e_i^\perp), Y \rangle.$$

From (3.42), (3.45), we get the second equation of (3.33).  $\square$

**Theorem 3.4.** *The following identity holds,*

$$(3.46) \quad \mathcal{O}_1 = -\frac{2}{3} (\partial_j R^{LB})_{x_0} (\mathcal{R}, e_i) Z_j \nabla_{0, e_i} - \frac{1}{3} (\partial_i R^{LB})_{x_0} (\mathcal{R}, e_i) \\ - 2 \langle A(e_i^0) e_j^0, \mathcal{R}^\perp \rangle \nabla_{0, e_i^0} \nabla_{0, e_j^0} - \pi \sqrt{-1} \langle (\nabla_{\mathcal{R}}^X \mathbf{J}) e_a, e_b \rangle c(e_a) c(e_b) \\ + 4\pi^2 \langle (\nabla_{\mathcal{R}^0}^X \mathbf{J}) (\mathcal{R}^0 + 2\mathcal{R}^\perp) + (\nabla_{\mathcal{R}^\perp}^X \mathbf{J}) \mathcal{R}^\perp - T(\mathcal{R}^\perp, \mathbf{J} \mathcal{R}^\perp), \mathbf{J} \mathcal{R}^\perp \rangle \\ + 4\pi^2 \left\langle \mathbf{J} A(\mathcal{R}^0) \mathcal{R}^0 + \frac{1}{2} T(\mathcal{R}^0, \mathbf{J} \mathcal{R}^0), \mathbf{J} \mathcal{R}^\perp \right\rangle + 4\pi \sqrt{-1} \langle \tilde{\mu}^{\text{Cliff}} + \tilde{\mu}^E, \mathbf{J} \mathcal{R}^\perp \rangle.$$

*Proof.* For  $\psi \in (T^*X \otimes \text{End}(\Lambda(T^{*(0,1)}X)))_B \simeq (T^*X \otimes (C(TX) \otimes_{\mathbb{R}} \mathbb{C}))_B$ , where  $C(TX)$  is the Clifford bundle of  $TX$ , we denote by  $\nabla^X \psi$  the covariant derivative of  $\psi$  induced by  $\nabla^{TX}$ .

From  $[\nabla_W^{\text{Cliff}}, c(e_a)] = c(\nabla_W^{TX} e_a)$ , we observe that for  $W \in TB$ ,

$$(3.47) \quad \begin{aligned} \nabla_W^X(\psi(e_a)c(e_a)) &= (\nabla_W^X \psi)(e_a)c(e_a) + \psi(\nabla_W^{TX} e_a)c(e_a) + \psi(e_a)c(\nabla_W^{TX} e_a) \\ &= (\nabla_W^X \psi)(e_a)c(e_a). \end{aligned}$$

Thus by (2.48) and (3.47), for  $k \geq 2$ ,

$$(3.48) \quad \begin{aligned} -(2\omega_d + \tau)(tZ) &= \frac{1}{2} (R^L(e_a, e_b) c(e_a) c(e_b)) (tZ) \\ &= \frac{1}{2} \sum_{r=0}^k \frac{\partial^r}{\partial t^r} [(R^L(e_a, e_b) c(e_a) c(e_b))(tZ)] \Big|_{t=0} \frac{t^r}{r!} + \mathcal{O}(t^{k+1}) \\ &= \frac{1}{2} \left( R_{x_0}^L + t(\nabla_{\mathcal{R}}^X R^L)_{x_0} \right) (e_a, e_b) c(e_a) c(e_b) + \mathcal{O}(t^2). \end{aligned}$$

By Lemma 3.3 and (2.111), we have

$$(3.49) \quad \begin{aligned} -t^2 \langle \tilde{\mu}^{E_p}, \tilde{\mu}^{E_p} \rangle (tZ) &= 4\pi^2 \sum_{k=2}^3 \frac{1}{k!} \frac{\partial^k}{\partial t^k} \left( |\tilde{\mu}|_{g^{TY}}^2(tZ) \right) \Big|_{t=0} t^{k-2} \\ &\quad + 4\pi \sqrt{-1} t \langle \tilde{\mu}^{\text{Cliff}} + \tilde{\mu}^E, \mathbf{J} \mathcal{R}^\perp \rangle_{x_0} + \mathcal{O}(t^2). \end{aligned}$$

The following two formulas are clear,

$$(3.50) \quad \begin{aligned} \frac{1}{2} \frac{\partial^2}{\partial t^2} |\tilde{\mu}|_{g^{TY}}^2(tZ) \Big|_{t=0} &= \frac{1}{2} (\nabla \nabla |\tilde{\mu}|_{g^{TY}}^2(Z))_{(\mathcal{R}, \mathcal{R})} \Big|_{Z=0} = |\nabla_{\mathcal{R}}^{TY} \tilde{\mu}|^2, \\ \frac{1}{3!} \frac{\partial^3}{\partial t^3} |\tilde{\mu}|_{g^{TY}}^2(tZ) \Big|_{t=0} &= \frac{1}{6} (\nabla \nabla \nabla |\tilde{\mu}|_{g^{TY}}^2(Z))_{(\mathcal{R}, \mathcal{R}, \mathcal{R})} \Big|_{Z=0} \\ &= \langle (\nabla^{TY} \nabla^{TY} \tilde{\mu})_{(\mathcal{R}, \mathcal{R})}, \nabla_{\mathcal{R}}^{TY} \tilde{\mu} \rangle. \end{aligned}$$

From Lemma 3.3 and (3.49)-(3.50), we see that the contribution from  $-t^2 \langle \tilde{\mu}^{E_p}, \tilde{\mu}^{E_p} \rangle (tZ)$  is the last three terms of (3.46).

By (2.104), (2.106) and (3.10), we have

$$(3.51) \quad \nabla_{t, e_i} = \nabla_{0, e_i} + \frac{t}{3} (\partial_j R^{LB})_{x_0} Z_j(\mathcal{R}, e_i) - \frac{t}{2} \left( \frac{1}{\kappa} \nabla_{e_i} \kappa \right) (tZ) + \mathcal{O}(t^2).$$

By  $g_{ij}(Z) = \theta_i^k(Z) \theta_j^k(Z)$  and (2.95)-(2.97), we know

$$(3.52) \quad \begin{aligned} g_{ij}(Z) &= \delta_{ij} - 2 \langle A(e_i^0) e_j^0, \mathcal{R}^\perp \rangle + \mathcal{O}(|Z|^2) \quad \text{for } 1 \leq i, j \leq 2(n - n_0), \\ &\delta_{ij} + \mathcal{O}(|Z|^2) \quad \text{otherwise;} \\ \kappa(Z) &= \det(g_{ij}(Z))^{1/2} = 1 - \langle A(e_i^0) e_i^0, \mathcal{R}^\perp \rangle + \mathcal{O}(|Z|^2). \end{aligned}$$

From (3.41), (3.51) and (3.52), the first three terms of the right hand side of (3.46) is the coefficient  $t^1$  of the Taylor expansion of  $-g^{ij}(tZ)(\nabla_{t, e_i} \nabla_{t, e_j} - t \nabla_{t, \nabla_{e_i}^{TB} e_j}(tZ))$ .

By (2.110), (3.43) and the above argument, the proof of Theorem 3.4 is complete.  $\square$

**Theorem 3.5.** *We have the relation*

$$(3.53) \quad P^N \mathcal{O}_1 P^N = 0.$$

*Proof.* By (3.8) and (3.19),

$$(3.54) \quad \begin{aligned} b_i^+ P^N &= b_i^{\perp+} P^N = 0, \quad (b_i^\perp P^N)(Z, Z') = 2a_i^\perp Z_i^\perp P^N(Z, Z'), \\ (b_i P^N)(Z, Z') &= a_i(\bar{z}_i^0 - \bar{z}_i^0) P^N(Z, Z'). \end{aligned}$$

We learn from (3.54) that for any polynomial  $g(Z^\perp)$  in  $Z^\perp$ , we can write  $g(Z^\perp)P^N(Z, Z')$  as sums of  $g_{\beta^\perp}(b^\perp)^{\beta^\perp}P^N(Z, Z')$  with constants  $g_{\beta^\perp}$ . By Theorem 3.1,

$$(3.55) \quad P_{\mathcal{L}^\perp}(b^\perp)^{\alpha^\perp}g(Z^\perp)P^N = 0, \quad \text{for } |\alpha^\perp| > 0.$$

Let  $\{w_a\}$  be an orthonormal basis of  $(T^{(1,0)}X, g^{TX})$ .

Note that if  $f, g$  are two  $\mathbb{C}$ -linear forms, then

$$f(e_a)g(e_a) = f(w_a)g(\bar{w}_a) + f(\bar{w}_a)g(w_a).$$

Thus by Theorem 3.1, (2.8), (3.21) and (3.54),

$$(3.56) \quad \begin{aligned} P^N \langle (\nabla_{\mathcal{R}}^X \mathbf{J})e_a, e_b \rangle c(e_a) c(e_b) P^N &= -2P^N \langle (\nabla_{\mathcal{R}}^X \mathbf{J})w_a, \bar{w}_a \rangle P^N \\ &= -2 \langle (\nabla_{z^0 + \bar{z}^0}^X \mathbf{J})w_a, \bar{w}_a \rangle P^N = \sqrt{-1} \operatorname{Tr} |_{TX} [J(\nabla_{z^0 + \bar{z}^0}^X \mathbf{J})] P^N. \end{aligned}$$

By (3.8), (3.12), (3.21), (3.46), (3.54)-(3.56), we get

$$(3.57) \quad \begin{aligned} P^N \mathcal{O}_1 P^N &= P^N \left\{ \frac{2}{3} (\partial_{\mathcal{R}} R^{LB})_{x_0}(\mathcal{R}, \frac{\partial}{\partial \bar{z}_i^0}) b_i - \frac{1}{3} (\partial_{e_i^0} R^{LB})_{x_0}(\mathcal{R}, e_i^0) \right. \\ &\quad + \frac{1}{3} (\partial_{\mathcal{R}} R^{LB})_{x_0}(\mathcal{R}, e_j^\perp) b_j^\perp - \frac{1}{3} (\partial_{e_j^\perp} R^{LB})_{x_0}(\mathcal{R}^0, e_j^\perp) \\ &\quad \left. + \pi \operatorname{Tr} |_{TX} [J(\nabla_{\mathcal{R}^0}^X \mathbf{J})] + 8\pi^2 \langle (\nabla_{\mathcal{R}^0}^X \mathbf{J}) \mathcal{R}^\perp, \mathbf{J} \mathcal{R}^\perp \rangle \right\} P^N. \end{aligned}$$

By (3.9), (3.54) and (3.55),

$$(3.58) \quad P^N Z_j^\perp Z_k^\perp P^N = \frac{1}{2a_k^\perp} P^N Z_j^\perp b_k^\perp P^N = \frac{1}{2a_k^\perp} \delta_{jk} P^N.$$

For  $\psi$  a tensor on  $X_G$ , let  $\nabla^{X_G} \psi$  be the covariant derivative of  $\psi$  induced by the Levi-Civita connection  $\nabla^{TX_G}$ .

For  $U, V, W \in T_{x_0} X_G$ , by (3.2), (3.3) and (3.11), we have

$$(3.59) \quad (\partial_U R^{LB})_{x_0}(V, W) = -2\pi \sqrt{-1} \langle (\nabla_U^{X_G} \mathbf{J}_G) V, W \rangle = -2\pi \sqrt{-1} \langle (\nabla_U^X \mathbf{J}) V, W \rangle.$$

From (3.2), (3.5), we know that

$$(3.60) \quad \mathbf{J} e_j^\perp = \frac{a_j}{2\pi} J e_j^\perp.$$

By Theorem 3.1, (1.18), (2.7), (3.9), (3.44) and (3.54)-(3.60), we get

$$\begin{aligned}
 (3.61) \quad P^N \mathcal{O}_1 P^N &= P^N \left\{ -\frac{4\pi\sqrt{-1}}{3} \left[ 2 \left\langle (\nabla_{\mathcal{R}^0}^X \mathbf{J}) \frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_i^0} \right\rangle + \left\langle (\nabla_{\frac{\partial}{\partial z_i^0}}^X \mathbf{J}) \mathcal{R}^0, \frac{\partial}{\partial \bar{z}_i^0} \right\rangle \right. \right. \\
 &\quad \left. \left. - \left\langle (\nabla_{\frac{\partial}{\partial \bar{z}_i^0}}^X \mathbf{J}) \mathcal{R}^0, \frac{\partial}{\partial z_i^0} \right\rangle \right] + \pi \operatorname{Tr} |_{TX} [J(\nabla_{\mathcal{R}^0}^X \mathbf{J})] + 2\pi \left\langle (\nabla_{\mathcal{R}^0}^X \mathbf{J}) e_j^\perp, J e_j^\perp \right\rangle \right\} P^N \\
 &= \pi \left[ -4\sqrt{-1} \left\langle (\nabla_{z^0 + \bar{z}^0}^X \mathbf{J}) \frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_i^0} \right\rangle + \operatorname{Tr} |_{TX} [J(\nabla_{z^0 + \bar{z}^0}^X \mathbf{J})] - 2 \left\langle J(\nabla_{z^0 + \bar{z}^0}^X \mathbf{J}) e_j^\perp, e_j^\perp \right\rangle \right] P^N = 0.
 \end{aligned}$$

The proof of Theorem 3.5 is complete.  $\square$

From (3.32) and Theorem 3.5, we get the following general formula which will be used in Section 5,

$$\begin{aligned}
 (3.62) \quad P^{(2)} &= (\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{O}_1 (\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{O}_1 P^N - (\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{O}_2 P^N \\
 &\quad + P^N \mathcal{O}_1 (\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{O}_1 (\mathcal{L}_2^0)^{-1} P^{N^\perp} - P^N \mathcal{O}_2 (\mathcal{L}_2^0)^{-1} P^{N^\perp} \\
 &\quad + (\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{O}_1 P^N \mathcal{O}_1 (\mathcal{L}_2^0)^{-1} P^{N^\perp} - P^N \mathcal{O}_1 (\mathcal{L}_2^0)^{-2} P^{N^\perp} \mathcal{O}_1 P^N.
 \end{aligned}$$

**3.4. Example**  $(\mathbb{C}P^1, 2\omega_{FS})$ . Let  $\omega_{FS}$  be the Kähler form associated to the Fubini-Study metric  $g_{FS}^{T\mathbb{C}P^1}$  on  $\mathbb{C}P^1$ . We will use the metric  $g^{T\mathbb{C}P^1} = 2g_{FS}^{T\mathbb{C}P^1}$  on  $\mathbb{C}P^1$  in this Subsection.

Let  $L$  be the holomorphic line bundle  $\mathcal{O}(2)$  on  $\mathbb{C}P^1$ . Recall that  $\mathcal{O}(-1)$  is the tautological line bundle of  $\mathbb{C}P^1$ .

We will use the homogeneous coordinate  $(z_0, z_1) \in \mathbb{C}^2$  for  $\mathbb{C}P^1 \simeq (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*$ .

Denote by  $U_i = \{[z_0, z_1] \in \mathbb{C}P^1; z_i \neq 0\}$ ,  $(i = 0, 1)$ , the open subsets of  $\mathbb{C}P^1$ , and the two coordinate charts are defined by  $\phi_i : U_i \simeq \mathbb{C}$ ,  $\phi_i([z_0, z_1]) = \frac{z_j}{z_i}$ ,  $j \neq i$ .

For any  $i_0, i_1 \in \mathbb{N}$ ,  $z_0^{i_0} z_1^{i_1}$  is naturally identified to a holomorphic section of  $\mathcal{O}(-i_0 - i_1)^*$  on  $\mathbb{C}P^1$ . For any  $k \in \mathbb{N}$ , we have

$$(3.63) \quad H^0(\mathbb{C}P^1, \mathcal{O}(k)) = \mathbb{C}\{s_{k,i_0} := z_0^{i_0} z_1^{i_1}, i_0 + i_1 = k, \text{ and } i_0, i_1 \in \mathbb{N}\}.$$

On  $U_i$ , the trivialization of the line bundle  $L$  is defined by  $L \ni s \rightarrow s/z_i^2$ , here  $z_i^2$  is considered as a holomorphic section of  $\mathcal{O}(2)$ .

In the following, we will work on  $\mathbb{C}$  by using  $\phi_1 : U_1 \rightarrow \mathbb{C}$ . Then for  $z \in \mathbb{C}$ ,

$$(3.64) \quad \omega_{FS}(z) = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \log((1 + |z|^2)^{-1}) = \frac{\sqrt{-1}}{2\pi} \frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}.$$

Let  $h^L$  be the smooth Hermitian metric on  $L$  on  $\mathbb{C}P^1$  defined by for  $z \in \mathbb{C}$ ,

$$(3.65) \quad |s_{2,0}|_{h^L}^2(z) = (1 + |z|^2)^{-2}.$$

Let  $\nabla^L$  be the holomorphic Hermitian connection of  $(L, h^L)$  with its curvature  $R^L$ .

By (3.64) and (3.65), under our trivialization on  $\mathbb{C}$

$$\begin{aligned}
 (3.66) \quad \nabla^L &= \bar{\partial} + \partial + \partial \log(|s_{2,0}|_{h^L}^2), \\
 \frac{\sqrt{-1}}{2\pi} R^L &= \frac{\sqrt{-1}}{2\pi} \bar{\partial} \partial \log |s_{2,0}|_{h^L}^2 = 2\omega_{FS} =: \omega.
 \end{aligned}$$

Let  $K$  be the canonical basis of  $\operatorname{Lie} S^1 = \mathbb{R}$ , i.e. for  $t \in \mathbb{R}$ ,  $\exp(tK) = e^{2\pi\sqrt{-1}t} \in S^1$ .

We define an  $S^1$ -action on  $\mathbb{C}P^1$  by  $g \cdot [z_0, z_1] = [gz_0, z_1]$  for  $g \in S^1$ .

On our local coordinate,  $g \cdot z = gz$ , and the vector field  $K^{\mathbb{C}P^1}$  on  $\mathbb{C}P^1$  induced by  $K$  is

$$(3.67) \quad K^{\mathbb{C}P^1}(z) := \frac{\partial}{\partial t} \exp(-tK) \cdot z|_{t=0} = -2\pi\sqrt{-1} \left( z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right).$$

Set

$$\mu(K)([z_0, z_1]) = \frac{2|z_0|^2}{|z_0|^2 + |z_1|^2} - 1.$$

Then, on  $\mathbb{C}$ ,

$$(3.68) \quad \mu(K) = 2|z|^2(1 + |z|^2)^{-1} - 1.$$

By (3.64), (3.67) and (3.68), we verify easily that  $\mu$  is a moment map associated to the  $S^1$ -action on  $(\mathbb{C}P^1, \omega)$  in the sense of (2.15).

The Lie  $S^1$ -action on the sections of  $L$  defined by (2.14) induces a holomorphical  $S^1$ -action on  $L$ . In particular, from (3.66)-(3.68),

$$(3.69) \quad \frac{\partial}{\partial t} \exp(-tK) \cdot s_{2,j}|_{t=0} =: L_K s_{2,j} = 2\pi\sqrt{-1}(1-j)s_{2,j}.$$

By (3.69), the  $S^1$ -invariant sub-space of  $H^0(\mathbb{C}P^1, L^p)$  and  $\mu^{-1}(0)$  are

$$(3.70) \quad H^0(\mathbb{C}P^1, L^p)^{S^1} = \mathbb{C} s_{2p,p}, \quad \mu^{-1}(0) = \{z \in \mathbb{C}, |z| = 1\},$$

and  $S^1$  acts freely on  $\mu^{-1}(0)$ , thus  $(\mathbb{C}P^1)_{S^1} = \{\text{pt}\}$ .

Under our trivialization of  $L$ ,  $s_{2p,j} \in H^0(\mathbb{C}P^1, L^p)$  is the function  $z^j$ , and from (3.65),

$$(3.71) \quad \|s_{2p,j}\|_{L^2}^2 = \int_{\mathbb{C}} \frac{|z|^{2j}}{(1 + |z|^2)^{2p}} 2\omega_{FS} = \int_0^\infty \frac{2t^j dt}{(1+t)^{2p+2}} = \frac{2j!(2p-j)!}{(2p+1)!}.$$

Thus  $(\frac{(2p+1)!}{2(p!)^2})^{1/2} s_{2p,p}$  is an orthonormal basis of  $H^0(\mathbb{C}P^1, L^p)^{S^1}$ .

Let  $\bar{\partial}^{L^p*}$  be the formal adjoint of the Dolbeault operator  $\bar{\partial}^{L^p}$ . For  $p \geq 1$ , the spin<sup>c</sup> Dirac operator  $D_p$  in (2.12) and its kernel are given by

$$(3.72) \quad D_p = \sqrt{2} \left( \bar{\partial}^{L^p} + \bar{\partial}^{L^p*} \right), \quad \text{Ker } D_p = H^0(\mathbb{C}P^1, L^p).$$

Finally, by Def. 2.3, for  $p \geq 1$ , we get

$$(3.73) \quad \begin{aligned} P_p^G(z, z') &= \frac{(2p+1)!}{2(p!)^2} s_{2p,p}(z) \otimes s_{2p,p}(z')^*, \\ P_p^G(z, z) &= \frac{(2p+1)!}{2(p!)^2} |s_{2p,p}|_{h^{L^p}}^2(z) = \frac{(2p+1)!}{2(p!)^2} \frac{|z|^{2p}}{(1 + |z|^2)^{2p}}. \end{aligned}$$

Note that our trivialization by  $s_{2,0}$  is not unitary, thus we do not see directly the off-diagonal decay (0.14) from (3.73).

Here we will only verify that (3.73) is compatible with (0.13), (0.15) and (0.16).

Recall that Stirling's formula [38, (3.A.40)] tells us that as  $p \rightarrow +\infty$ ,

$$(3.74) \quad p! = (2\pi p)^{1/2} p^p e^{-p} \left( 1 + \mathcal{O}\left(\frac{1}{p}\right) \right).$$

By (3.74),

$$(3.75) \quad \frac{(2p+1)!}{2(p!)^2} = \frac{\sqrt{p}}{\sqrt{\pi} e} 2^{2p} \left( 1 + \frac{1}{2p} \right)^{2p} \left( 1 + \mathcal{O}\left(\frac{1}{p}\right) \right) = \sqrt{\frac{p}{\pi}} 2^{2p} \left( 1 + \mathcal{O}\left(\frac{1}{p}\right) \right).$$

Now,  $\mathbb{C}^*$  is an open neighborhood of  $\mu^{-1}(0)$  and  $B = \mathbb{C}^*/S^1 \simeq \mathbb{R}^+$  by mapping  $z \in \mathbb{C}^*$  to  $r = |z| \in \mathbb{R}^+$ .

By (3.64), the metrics on  $\{|z| = r\} = \{re^{2\pi\sqrt{-1}\theta}; \theta \in \mathbb{R}/\mathbb{Z}\}$ ,  $B \simeq \mathbb{R}^+$  induced by  $\omega = 2\omega_{FS}$  is

$$(3.76) \quad 8\pi r^2 (1+r^2)^{-2} d\theta \otimes d\theta, \quad g^{TB} = \frac{2}{\pi}(1+r^2)^{-2} dr \otimes dr.$$

From (3.76), the fiberwise volume function  $h^2(r)$  in (0.10) on  $\mathbb{R}^+$  is

$$(3.77) \quad h^2(r) = \sqrt{8\pi} r (1+r^2)^{-1}.$$

From (3.73), (3.75) and (3.77), we get for  $|z| = r$ ,

$$(3.78) \quad h^2(r)P_p^G(z, z) = \sqrt{8\pi} \frac{(2p+1)!}{2(p!)^2} \left(\frac{r}{1+r^2}\right)^{2p+1} = \sqrt{2p} \left(\frac{2r}{1+r^2}\right)^{2p+1} \left(1 + \mathcal{O}\left(\frac{1}{p}\right)\right).$$

When  $|z| = 1$ , from (3.78), we re-find (0.15) and (0.16).

From (3.76),  $\sqrt{2\pi} \frac{\partial}{\partial r}$  is an orthonormal basis of  $(B, g^{TB})$  at  $r = 1$ , thus the normal coordinate  $Z^\perp$  has the form  $r - 1 = \sqrt{2\pi}(Z^\perp + \mathcal{O}(|Z^\perp|^2))$ . Thus

$$(3.79) \quad (2r(1+r^2)^{-1})^{2p+1} = e^{(2p+1)\log(1-\pi(Z^\perp)^2 + \mathcal{O}(|Z^\perp|^3))} = e^{-2\pi p(Z^\perp)^2} + \dots$$

This means that (3.78), (3.79) are compatible with (0.13) and (3.22).

If we consider the sub-space  $H^0(\mathbb{CP}^1, L^p)_{-p}$  of  $H^0(\mathbb{CP}^1, L^p)$  with the weight  $-p$  of  $S^1$ -action, then by (2.14) as in (3.69), and (3.71),  $\sqrt{p + \frac{1}{2}} s_{2p,0}$  is an orthonormal basis of  $H^0(\mathbb{CP}^1, L^p)_{-p}$ .

Thus the smooth kernel  $P_p^{-p}(z, z')$  of the orthogonal projection from  $\mathcal{C}^\infty(\mathbb{CP}^1, L^p)$  onto  $H^0(\mathbb{CP}^1, L^p)_{-p}$  is

$$(3.80) \quad P_p^{-p}(z, z') = (p + \frac{1}{2}) s_{2p,0}(z) \otimes s_{2p,0}(z')^*, \quad P_p^{-p}(z, z) = (p + \frac{1}{2})(1 + |z|^2)^{-2p}.$$

Note that  $\mu^{-1}(-1) = \{0\}$ , i.e.  $-1$  is a singular value of  $\mu$ .

Let  $\mu_1$  be the moment map defined by  $\mu_1(K) = \mu(K) + 1$ , then  $H^0(\mathbb{CP}^1, L^p)_{-p}$  is the corresponding  $S^1$ -invariant holomorphic sections of  $L^p$  with respect to the corresponding  $S^1$ -action.

Thus 0 is a singular value of  $\mu_1$  and this explains why we have a factor  $p$  in (3.80) instead of  $p^{1/2}$  in (3.78).

#### 4. APPLICATIONS

This Section is organized as follows. In Section 4.1, we explain Theorem 4.1, the version of Theorem 0.2 when we only assume that  $\mu$  is regular at 0. In Section 4.2, We explain how to handle the  $\vartheta$ -weight Bergman kernel. In Section 4.3, we deduce (0.15), and (0.16) from [17, Theorem 4.18']. In Section 4.4, we explain Theorem 0.2 implies Toeplitz operator type properties on  $X_G$ . In Section 4.5, we extend our results for non-compact manifolds and for covering spaces. In Section 4.6, we explain the relation on the  $G$ -invariant Bergman kernel on  $X$  and the Bergman kernel on  $X_G$ .

We use the notation in Introduction.

**4.1. Orbifold case.** In this Subsection, we only suppose that  $0 \in \mathfrak{g}^*$  is a regular value of  $\mu$ , then  $G$  acts only infinitesimal freely on  $P = \mu^{-1}(0)$ , thus  $X_G = P/G$  is a compact symplectic orbifold.

Let  $G^0 = \{g \in G, g \cdot x = x \text{ for any } x \in P\}$ , then  $G^0$  is a finite normal sub-group of  $G$  and  $G/G_0$  acts effectively on  $P$ .

We will use the notation for the orbifold as in [24, §1], [17, §4.2].

Let  $U$  be a  $G$ -neighborhood of  $P = \mu^{-1}(0)$  in  $X$  such that  $G$  acts infinitesimal freely on  $\overline{U}$ , the closure of  $U$ . From the construction in Section 1.2, any  $G$ -equivariant vector bundle  $F$  on  $U$  induces an orbifold vector bundle  $F_B$  on the orbifold  $B = U/G$ .

The function  $h$  in (0.10) is only  $\mathcal{C}^\infty$  on the regular part of the orbifold  $B$ , and we extend continuously  $h$  to  $U/G$  from its regular part, which is  $\mathcal{C}^\infty$  and we denote it by  $\widehat{h}$ , then  $\widehat{h}$  is also  $\mathcal{C}^\infty$  on  $U$ .

As we work on  $P$  in Section 2.5, we need not to modify this part.

We need to modify Section 2.6 as follows.

Observe first that the construction in Section 1.1 works well if we only assume that  $G$  acts locally freely on  $X$  therein.

Denote by  $\nabla^{T^H U}$  the connection on  $T^H U$  as in Section 1.1, and on  $P$ , let  $\nabla^N, \nabla^{T^H P}, {}^0\nabla^{T^H U}$  be the connections on  $N, T^H U$  in Section 2.5 as in (0.9).

For  $y_0 \in P$ ,  $W \in T^H U$  (resp.  $T^H P$ ), we define  $\mathbb{R} \ni t \rightarrow x_t = \exp_{y_0}^{T^H U}(tW) \in U$  (resp.  $\exp_{y_0}^{T^H P}(tW) \in P$ ) the curve such that  $x_t|_{t=0} = y_0$ ,  $\frac{dx}{dt}|_{t=0} = W$ ,  $\frac{dx}{dt} \in T^H U$ ,  $\nabla_{\frac{dx}{dt}}^{T^H U} \frac{dx}{dt} = 0$  (resp.  $\frac{dx}{dt} \in T^H P$ ,  $\nabla_{\frac{dx}{dt}}^{T^H P} \frac{dx}{dt} = 0$ ).

By proceeding as in Section 2.6, we identify  $B^{T^H U}(y_0, \varepsilon)$  to a subset of  $U$  as following, for  $Z \in B^{T^H U}(y_0, \varepsilon)$ ,  $Z = Z^0 + Z^\perp$ ,  $Z^0 \in T_{y_0}^H P$ ,  $Z^\perp \in N_{x_0}$ , we identify  $Z$  with  $\exp_{\exp_{x_0}^{T^H P}(Z^0)}^{T^H U}(\tau_{Z^0} Z^\perp)$ .

Set  $G_{y_0} = \{g \in G, gy_0 = y_0\}$ , then  $G \cdot B^{T^H U}(y_0, \varepsilon) = G \times_{G_{y_0}} B^{T^H U}(y_0, \varepsilon)$  is a  $G$ -neighborhood of  $Gy_0$ , and  $(G_{y_0}, B^{T^H U}(y_0, \varepsilon))$  is a local coordinate of  $B$ .

As the construction in Section 2.6 is  $G_{y_0}$ -equivariant, we extend the geometric objects on  $G \times_{G_{y_0}} B^{T^H U}(y_0, \varepsilon)$  to  $G \times_{G_{y_0}} \mathbb{R}^{2n-n_0} = X_0$ .

Thus we get the corresponding geometric objects on  $G \times \mathbb{R}^{2n-n_0}$  by using the covering  $G \times \mathbb{R}^{2n-n_0} \rightarrow G \times_{G_{y_0}} \mathbb{R}^{2n-n_0}$ , especially,  $\widehat{\mathcal{L}}_p^{X_0}$  (where we use the  $\widehat{\cdot}$  notation to indicate the modification) is defined similarly on  $G \times \mathbb{R}^{2n-n_0}$ , and Theorem 2.5 holds for  $\widehat{\mathcal{L}}_p^{X_0}$ .

Let  $\widehat{\pi}_G : G \times \mathbb{R}^{2n-n_0} \rightarrow \mathbb{R}^{2n-n_0}$  be the natural projection and as in (1.20), we define  $\widehat{\Phi} = \widehat{h}\widehat{\pi}_G$ , then the operator  $\widehat{\Phi}\widehat{\mathcal{L}}_p^{X_0}\widehat{\Phi}^{-1}$  is well-defined on  $T_{y_0}^H U \simeq \mathbb{R}^{2n-n_0}$ .

Let  $g^{T^H X_0}$  be the metric on  $\mathbb{R}^{2n-n_0}$  induced by  $g^{T^H X_0}$ , and let  $dv_{T^H X_0}$  be the Riemannian volume form on  $(\mathbb{R}^{2n-n_0}, g^{T^H X_0})$ .

Let  $P_{y_0,p}$  be the orthogonal projection from  $L^2(\mathbb{R}^{2n-n_0}, (\Lambda(T^{*(0,1)} X) \otimes L^p \otimes E)_{y_0})$  onto  $\text{Ker}(\widehat{\Phi}\widehat{\mathcal{L}}_p^{X_0}\widehat{\Phi}^{-1})$  on  $\mathbb{R}^{2n-n_0}$ . Let  $P_{y_0,p}(Z, Z')$  ( $Z, Z' \in \mathbb{R}^{2n-n_0}$ ) be the smooth kernel of  $P_{y_0,p}$  with respect to  $dv_{T^H X_0}(Z')$ .

Let  $P_{0,p}^G$  be the orthogonal projection from  $\Omega^{0,\bullet}(X_0, L_0^p \otimes E_0)$  on  $(\text{Ker } D_p^{X_0})^G$ , and let  $P_{0,p}^G(x, x')$  be the smooth kernel of  $P_{0,p}^G$  with respect to the volume form  $dv_{X_0}(x')$ .

Let  $P_p^{X_0/G}(y, y')$  ( $y, y' \in X_0/G$ ) be the smooth kernel associated to the operator on  $X_0/G$  induced by  $\widehat{\Phi}\widehat{\mathcal{L}}_p^{X_0}\widehat{\Phi}^{-1}$  as in (2.72).

Note that our trivialization of the restriction of  $L$  on  $B^{T^H U}(y_0, \varepsilon)$  as in Section 2.6 is not  $G_{y_0}$ -invariant, except that  $G_{y_0}$  acts trivially on  $L_{y_0}$ .

For  $x, x' \in X_0$ , with their representatives  $\tilde{x}, \tilde{x}' \in \mathbb{R}^{2n-n_0}$ , we have

$$(4.1) \quad \widehat{h}(x)\widehat{h}(x')P_{0,p}^G(x, x') = P_p^{X_0/G}(\pi(x), \pi(x')) = \frac{1}{|G^0|} \sum_{g \in G_{y_0}} (g, 1) \cdot P_{y_0,p}(g^{-1}\tilde{x}, \tilde{x}').$$

The second equation of (4.1) is from direct computation (cf. [17, (5.19)]).

As we work on  $G \times \mathbb{R}^{2n-n_0}$ , for the operator  $\widehat{\Phi}\widehat{\mathcal{L}}_p^{X_0}\widehat{\Phi}^{-1}$ , Prop. 2.9 and Sections 2.7-2.9 still holds.

From Theorem 2.23 for  $P_{y_0,p}$  and (4.1), we get

**Theorem 4.1.** *Under the same notation in Theorems 0.2, 2.23, for  $\alpha, \alpha' \in \mathbb{N}^{2n-n_0}$ ,  $|\alpha| + |\alpha'| \leq m$ , we have*

$$(4.2) \quad (1 + \sqrt{p}|Z^\perp| + \sqrt{p}|Z'^\perp|)^{m''} \left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left( p^{-n+\frac{n_0}{2}} (\widehat{h}\kappa^{\frac{1}{2}})(Z) (\widehat{h}\kappa^{\frac{1}{2}})(Z') P_p^G \circ \Psi(Z, Z') \right. \right. \\ \left. \left. - \frac{1}{|G^0|} \sum_{r=0}^k \sum_{g \in G_{y_0}} (g, 1) \cdot P_{y_0}^{(r)}(g^{-1}\sqrt{p}Z, \sqrt{p}Z') p^{-\frac{r}{2}} \right) \right|_{\mathcal{C}^{m'}(P)} \\ \leq Cp^{-(k+1-m)/2} (1 + \sqrt{p}|Z^0| + \sqrt{p}|Z'^0|)^{2(n+k+m'+2)+m} \exp(-\sqrt{C''\nu p} \inf_{g \in G_{y_0}} |g^{-1}Z - Z'|) \\ + \mathcal{O}(p^{-\infty}).$$

If  $Z = Z' = Z^0$ , then for  $g \in G_{y_0}$ ,  $gZ^0 = Z^0$ , we use Theorem 2.23 for  $Z = Z' = 0$  with the base point  $Z^0$ , and for the rest element in  $G_{y_0}$ , we use Theorem 2.23 for  $Z = Z' = Z^0$  with the base point  $y_0$ , then we get

$$(4.3) \quad \left| p^{-n+\frac{n_0}{2}} (\widehat{h}^2\kappa)(Z^0) P_p^G \circ \Psi(Z^0, Z^0) - \frac{1}{|G^0|} \sum_{r=0}^k \sum_{g \in G_{y_0}, gZ^0=Z^0} (g, 1) \cdot P_{Z^0}^{(2r)}(0, 0) p^{-r} \right. \\ \left. - \frac{1}{|G^0|} \sum_{r=0}^{2k} \sum_{g \in G_{y_0}, gZ^0 \neq Z^0} (g, 1) \cdot P_{y_0}^{(r)}(g^{-1}\sqrt{p}Z^0, \sqrt{p}Z^0) p^{-\frac{r}{2}} \right| \\ \leq Cp^{-(2k+1)/2} \left( 1 + (1 + \sqrt{p}|Z^0|)^{2(n+2k+2)} \exp(-\sqrt{C''' \nu p} |Z^0|) \right).$$

Note that if  $g \in G_{y_0}$  acts as the multiplication by  $e^{i\theta}$  on  $L_{y_0}$ , then  $(g, 1) \cdot P_{y_0}^{(r)}$ ,  $(g, 1) \cdot P_{Z^0}^{(r)}$  in (4.3) have a factor  $e^{i\theta p}$  which depends on  $p$ .

Of course, after replacing  $L$  by some power of  $L$ , we can assume that  $G_{y_0}$  acts as identity on  $L$  for any  $y_0 \in P$ , in this case,  $(g, 1) \cdot P_{y_0}^{(r)}$ ,  $(g, 1) \cdot P_{Z^0}^{(r)}$  do not depend on  $p$ .

From Theorem 3.2 and (4.3), if the singular set of  $X_G$  is not empty, analogous to the usual orbifold case [17, (5.27)],  $p^{-n+\frac{n_0}{2}} P_p^G(y_0, y_0)$ , ( $y_0 \in P$ ) does not have a uniform asymptotic expansion in the form  $\sum_{r=0}^{\infty} c_r(y_0) p^{-r}$ .



**4.2.  $\vartheta$ -weight Bergman kernel on  $X$ .** In this section, we assume that  $G$  acts on  $P = \mu^{-1}(0)$  freely.

Let  $\mathcal{V}$  be a finite dimensional irreducible representation of  $G$ , we denote it by  $\rho^\mathcal{V} : G \rightarrow \text{End}(\mathcal{V})$ . Let  $\vartheta$  be the highest weight of the representation  $\mathcal{V}$ . Let  $\mathcal{V}^*$  be the trivial bundle on  $X$  with  $G$ -action  $\rho^{\mathcal{V}^*}$  induced by  $\rho^\mathcal{V}$ .

Let  $P_p^\mathcal{V}$  be the orthogonal projection from  $\Omega^{0,\bullet}(X, L^p \otimes E)$  on  $\text{Hom}_G(\mathcal{V}, \text{Ker } D_p) \otimes \mathcal{V} \subset \text{Ker } D_p$ . Let  $P_p^\mathcal{V}(x, x')$ ,  $(x, x' \in X)$ , be the smooth kernel of  $P_p^\mathcal{V}$  with respect to  $dv_X(x')$ .

We call  $P_p^\mathcal{V}(x, x')$  the  $\vartheta$ -weight Bergman kernel of  $D_p$ .

We explain now the asymptotic expansion of  $P_p^\mathcal{V}(x, x')$  as  $p \rightarrow \infty$ .

We will consider the corresponding objects in Sections 1-3 by replacing  $E$  by  $E \otimes \mathcal{V}^*$ . Especially, we denote by  $D_p^{\mathcal{V}^*}$  the corresponding spin<sup>c</sup> Dirac operator associated to the bundle  $L^p \otimes E \otimes \mathcal{V}^*$ .

Certainly, all results in Sections 1-3 still hold for the bundle  $E \otimes \mathcal{V}^*$ .

Let  $P_p^\vartheta$  be the orthogonal projection from  $\mathcal{C}^\infty(X, E_p \otimes \mathcal{V}^*)$  onto  $(\text{Ker } D_p^{\mathcal{V}^*})^G$ , and  $P_p^\vartheta(x, x')$ ,  $(x, x' \in X)$  the smooth kernel of  $P_p^\vartheta$  with respect to  $dv_X(x')$ .

As  $\mathcal{V}$  is an irreducible representation of  $G$ , we get

$$(4.4) \quad \text{Ker } D_p^{\mathcal{V}^*} = (\text{Ker } D_p) \otimes \mathcal{V}^*, \quad (\text{Ker } D_p^{\mathcal{V}^*})^G = \text{Hom}_G(\mathcal{V}, \text{Ker } D_p).$$

Let  $\{v_i\}$  be an orthonormal basis of  $\mathcal{V}$  with respect to a  $G$ -invariant metric on  $\mathcal{V}$  and  $\{v_i^*\}$  the corresponding dual basis.

Let  $dg$  be a Haar measure on  $G$ . By Schur Lemma,

$$(4.5) \quad \int_G g \cdot (v_j \otimes v_i^*) dg = \frac{1}{\dim_{\mathbb{C}} \mathcal{V}} \delta_{ij} \text{Id}_{\mathcal{V}}.$$

Thus if  $W$  is a finite dimensional representation of  $G$  with the highest weight  $\vartheta$ , then for any  $s \in W$ , we have

$$(4.6) \quad s = (\dim_{\mathbb{C}} \mathcal{V}) \left( \int_G g \cdot (s \otimes v_i^*) dg \right) \otimes v_i \in \text{Hom}_G(\mathcal{V}, W) \otimes \mathcal{V} = W.$$

From (4.6) and the  $G \times G$ -invariance of the kernel  $P_p^\vartheta(x, x')$ , we get

$$(4.7) \quad \begin{aligned} P_p^\mathcal{V}(x, x') &= (\dim_{\mathbb{C}} \mathcal{V}) \sum_i (P_p^\vartheta(x, x') v_i^*, v_i), \\ P_p^\mathcal{V}(x, x) &= (\dim_{\mathbb{C}} \mathcal{V}) \text{Tr}_{\mathcal{V}^*} P_p^\vartheta(x, x) \in \text{End}(\Lambda(T^{*(0,1)} X) \otimes E)_x. \end{aligned}$$

In fact, let  $\{\psi_j\}$  be an orthonormal basis of  $\text{Ker}(D_p^{\mathcal{V}^*})^G$ , then for any  $j$  fixed, in view of the second equality in (4.4), one sees that

$$(4.8) \quad \psi_j^* \psi_j \in \text{End}_G(\mathcal{V}) \quad \text{and} \quad \text{Tr}_{\mathcal{V}}[\psi_j^* \psi_j] = \|\psi_j\|_{L^2}^2 = 1.$$

Thus by Schur Lemma,

$$(4.9) \quad \psi_j^* \psi_j = \frac{1}{\dim_{\mathbb{C}} \mathcal{V}} \text{Id}_{\mathcal{V}}$$

and  $\{(\dim_{\mathbb{C}} \mathcal{V})^{\frac{1}{2}} \psi_j v_i\}$  is an orthonormal basis of  $\text{Hom}_G(\mathcal{V}, \text{Ker } D_p) \otimes \mathcal{V} \subset \text{Ker } D_p$ .

Let  $U$  be a  $G$ -neighborhood of  $P = \mu^{-1}(0)$  as in Theorem 0.2,  $P_p^\vartheta$  is viewed as a smooth section of  $\mathrm{pr}_1^*(E_p \otimes \mathcal{V}^*)_B \otimes \mathrm{pr}_2^*(E_p \otimes \mathcal{V}^*)_B^*$  on  $B \times B$ , or as a  $G \times G$ -invariant smooth section of  $\mathrm{pr}_1^*(E_p \otimes \mathcal{V}^*) \otimes \mathrm{pr}_2^*(E_p \otimes \mathcal{V}^*)^*$  on  $U \times U$ .

Moreover,  $v_i, v_i^*$  are smooth (not  $G$ -invariant) sections of  $U \times \mathcal{V}, U \times \mathcal{V}^*$  on  $U$ . Thus from (4.7),  $P_p^\mathcal{V}$  is not a  $G \times G$ -invariant section of  $\mathrm{pr}_1^*(E_p) \otimes \mathrm{pr}_2^*(E_p^*)$  on  $U \times U$ .

Now Theorem 0.2 applies well to the bundle  $E \otimes \mathcal{V}^*$ , thus we get the asymptotic expansion of  $P_p^\vartheta(x, x')$  as  $p \rightarrow +\infty$ , and the leading term in the expansion of  $p^{-n+\frac{n_0}{2}}(h\kappa^{\frac{1}{2}})(x)(h\kappa^{\frac{1}{2}})(x')P_p^\vartheta(x, x')$  is  $P(\sqrt{p}Z, \sqrt{p}Z')I_{\mathbb{C} \otimes (E \otimes \mathcal{V}^*)_B}$ .

By (4.7), the leading term of the asymptotic expansion of  $p^{-n+\frac{n_0}{2}}(h\kappa^{\frac{1}{2}})(x)(h\kappa^{\frac{1}{2}})(x')P_p^\mathcal{V}(x, x')$  is

$$(4.10) \quad (\dim_{\mathbb{C}} \mathcal{V})^2 P(\sqrt{p}Z, \sqrt{p}Z')I_{\mathbb{C} \otimes E_B}, \quad P(0, 0) = 2^{n_0/2}.$$

Let  $\Theta$  be the curvature of  $P \rightarrow X_G$  as in Section 1.1. Let  $\rho_*^{\mathcal{V}^*}$  denote the differential of  $\rho^{\mathcal{V}^*}$ . By (1.18),

$$(4.11) \quad R^{(E \otimes \mathcal{V}^*)G} = R^{E_G} + \rho_*^{\mathcal{V}^*}(\Theta).$$

In the same way, we can define  $\mathcal{S}_p^\mathcal{V}$  a section of  $\mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_B$  on  $X_G$  by (0.17) for  $P_p^\mathcal{V}$ . From (0.25) (which will be proved in Section 5), (4.7), (4.10) and (4.11), we get

**Theorem 4.2.** *Under the condition of Theorem 0.6, the first coefficients of the asymptotic expansion of  $\mathcal{S}_p^\mathcal{V} \in \mathrm{End}(E_G)$  in (0.20) is*

$$(4.12) \quad \begin{aligned} \Phi_0 &= (\dim_{\mathbb{C}} \mathcal{V})^2, \\ \Phi_1 &= \frac{1}{8\pi} (\dim_{\mathbb{C}} \mathcal{V})^2 \left( r_{x_0}^{X_G} + 6\Delta_{X_G} \log h + 4R_{x_0}^{E_G}(w_j^0, \overline{w}_j^0) \right) \\ &\quad + \frac{1}{2\pi} (\dim_{\mathbb{C}} \mathcal{V}) \mathrm{Tr}_{\mathcal{V}^*} [\rho_*^{\mathcal{V}^*}(\Theta)(w_j^0, \overline{w}_j^0)]. \end{aligned}$$

**4.3. Averaging the Bergman kernel: a direct proof of (0.15) and (0.16).** We use the same assumption and notation in Theorem 0.2.

Let  $P_p(x, x')$  be the smooth kernel of the orthogonal projection  $P_p$  from  $\Omega^{0,\bullet}(X, L^p \otimes E)$  onto  $\mathrm{Ker} D_p$  with respect to  $dv_X(x')$ . Then  $P_p(x, x')$  is the usual Bergman kernel associated to  $D_p$ .

Let  $dg$  be a Haar measure on  $G$ . By Schur Lemma,

$$(4.13) \quad P_p^G(x, x') = \int_G ((g, 1) \cdot P_p)(x, x') dg = \int_G (g, 1) \cdot P_p(g^{-1}x, x') dg.$$

One possible way to get Theorem 0.2 is to apply the full off-diagonal expansion [17, Theorem 4.18'] to (4.13).

Unfortunately, we do not know how to get the full off-diagonal expansion, especially the fast decay along  $N_G$  in (0.14) in this way.

However, it is easy to get (0.15) and (0.16) as direct consequences of [17, Theorem 4.18'] and (4.13).

As in Section 2.5, we denote by  $TY$  the sub-bundle of  $TX$  on a neighborhood of  $P = \mu^{-1}(0)$  generated by the  $G$ -action and by  $T^H P$  the orthogonal complement of  $TY$  in  $(TP, g^{TP})$ .

Take  $y_0 \in P$ . Let  $\{e_i\}_{i=1}^{2(n-n_0)}$ ,  $\{f_l\}_{l=1}^{n_0}$  be orthonormal basis of  $T_{y_0}^H P$ ,  $T_{y_0} Y$ . Then  $\{e_i\}_{i=1}^{2(n-n_0)} \cup \{f_l, J_{y_0} f_l\}_{l=1}^{n_0}$  is an orthonormal basis of  $T_{y_0} X$ . We use this orthonormal basis to get a local coordinate of  $X$  by using the exponential map  $\exp_{y_0}^X$ .

We identify  $B^{T_{y_0} X}(0, \varepsilon)$  to  $B^X(y_0, \varepsilon)$  by the exponential map  $Z \rightarrow \exp_{y_0}^X(uZ)$ .

Let  $\nabla^{\text{Cliff} \otimes E}$  be the connection on  $\Lambda(T^{*(0,1)} X) \otimes E$  induced by  $\nabla^{\text{Cliff}}$  and  $\nabla^E$ .

For  $Z \in B^{T_{y_0} X}(0, \varepsilon)$ , we identify  $L_Z$ ,  $(\Lambda(T^{*(0,1)} X) \otimes E)_Z$ ,  $(E_p)_Z$  to  $L_{y_0}$ ,  $(\Lambda(T^{*(0,1)} X) \otimes E)_{y_0}$ ,  $(E_p)_{y_0}$  by parallel transport with respect to the connections  $\nabla^L$ ,  $\nabla^{\text{Cliff} \otimes E}$ ,  $\nabla^{E_p}$  along the curve  $\gamma_Z : [0, 1] \ni u \rightarrow uZ$ .

Under this identification, for  $Z, Z' \in B^{T_{y_0} X}(0, \varepsilon)$ , one has

$$P_p(Z, Z') \in \text{End}(\Lambda(T^{*(0,1)} X) \otimes E)_{y_0}.$$

Let  $\kappa_1(Z)$  be the function on  $B^{T_{y_0} X}(0, \varepsilon)$  defined by

$$(4.14) \quad dv_X(Z) = \kappa_1(Z) dv_{T_{x_0} X}.$$

By [17, Theorem 4.18'] (i.e. Theorem 0.2 for  $G = \{1\}$ ), there exist  $J_r(Z') \in \text{End}(\Lambda(T^{*(0,1)} X) \otimes E)_{y_0}$ , polynomials in  $Z'$  with the same parity as  $r$ , such that for any  $k, m' \in \mathbb{N}$ , there exist  $C, M > 0$  such that for  $Z' \in T_{y_0} X$ ,  $|Z'| \leq \varepsilon$ ,

$$(4.15) \quad \left| \frac{1}{p^n} P_p(Z', 0) - \sum_{r=0}^k J_r(\sqrt{p} Z') \kappa_1^{-1}(Z') e^{-\frac{\pi}{2} p |Z'|^2} p^{-\frac{r}{2}} \right|_{\mathcal{C}^{m'}(P)} \leq C p^{-(k+1)/2} (1 + \sqrt{p} |Z'|)^M \exp(-\sqrt{C''} \nu \sqrt{p} |Z'|) + \mathcal{O}(p^{-\infty}),$$

and

$$(4.16) \quad J_0(Z) = I_{\mathbb{C} \otimes E}.$$

For  $K \in \mathfrak{g}$ ,  $|K|$  small,  $e^K$  maps  $(\Lambda(T^{*(0,1)} X) \otimes E)_{e^{-K} y_0}$ ,  $L_{e^{-K} y_0}$  to  $(\Lambda(T^{*(0,1)} X) \otimes E)_{y_0}$ ,  $L_{y_0}$ , and under our identification, we denote these maps by

$$(4.17) \quad f^E(K) \in \text{End}(\Lambda(T^{*(0,1)} X) \otimes E)_{y_0}, \quad f^L(K) \in \text{End}(L_{y_0}) \simeq \mathbb{C}.$$

By [1, Prop. 5.1], if we denote by

$$(4.18) \quad j_{\mathfrak{g}}(K) = \det_{\mathfrak{g}} \left( \frac{1 - e^{-ad K}}{ad K} \right)$$

for  $K \in \mathfrak{g}$ , then in exponential coordinates of  $G$ ,

$$(4.19) \quad d(e^K) = j_{\mathfrak{g}}(K) dK.$$

By [17, Prop. 4.1] (i.e. Theorem 0.1 for  $G = \{1\}$ ), (4.13), as  $G$  acts freely on  $P$ , we know

$$(4.20) \quad P_p^G(y_0, y_0) = \int_{K \in \mathfrak{g}, |K| \leq \varepsilon} f^E(K) (f^L(K))^p P_p(e^{-K} y_0, y_0) j_{\mathfrak{g}}(K) dK + \mathcal{O}(p^{-\infty}).$$

Let  $S^L$  be the section of  $L$  on  $B^{T_{y_0}X}(0, \varepsilon)$  obtained by parallel transport of a unit vector of  $L_{y_0}$  with respect to the connection  $\nabla^L$  along the curve  $\gamma_Z$ . Let  $\Gamma^L$  be the connection form of  $L$  with respect to this trivialization.

Recall that for  $K \in \mathfrak{g}$ , the corresponding vector field  $K^X$  on  $X$  is defined in Section 1.1. Recall that  $\{K_i\}$  is a basis of  $\mathfrak{g}$ .

By (2.105), for  $K \in \mathfrak{g}$ ,

$$(4.21) \quad \begin{aligned} (e^K \cdot S^L)(0) &= e^K \cdot S^L(e^{-K}y_0) = f^L(K)S^L(0), \text{ with } f^L(0) = 1, \\ \Gamma_Z^L(K^X) &= \frac{1}{2}R_{y_0}^L(Z, K^X) + \mathcal{O}(|Z|^2). \end{aligned}$$

By (2.14), (2.15), (4.21) and  $\mu = 0$  on  $P$ , we get

$$(4.22) \quad \begin{aligned} (L_{K_j}(L_{K_i}S^L))(0) &= (\nabla_{K_j^X}^L(\nabla_{K_i^X}^L S^L - 2\pi\sqrt{-1}\mu(K_i)S^L))(0) \\ &= \frac{1}{2}R_{y_0}^L(K_j^X, K_i^X)S^L(0) = \pi\sqrt{-1}\langle d\mu(K_i), K_j^X \rangle S^L(0) = 0. \end{aligned}$$

By (2.14), (4.21), (4.22) and  $\mu = 0$  on  $P$ , we get

$$(4.23) \quad \begin{aligned} \frac{\partial f^L}{\partial K_i}(0)S^L(0) &= (L_{K_i}S^L)(0) = (\nabla_{K_i^X}^L S^L)(0) = 0, \\ \frac{\partial^2 f^L}{\partial K_i \partial K_j}(0)S^L(0) &= \frac{\partial^2}{\partial t_1 \partial t_2}(e^{t_1 K_i + t_2 K_j} \cdot S^L)(0)|_{t_1=t_2=0} \\ &= (L_{K_j}(L_{K_i}S^L) + L_{K_i}(L_{K_j}S^L))(0) = 0. \end{aligned}$$

Thus from (4.23),

$$(4.24) \quad (f^L(K))^p = (1 + \mathcal{O}(|K|^3))^p.$$

Moreover,

$$(4.25) \quad f^E(K) = \text{Id}_{(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}} + \mathcal{O}(|K|), \quad \kappa_1(Z) = 1 + \mathcal{O}(|Z|^2).$$

Let  $dv_Y$  be the Riemannian volume form on  $(TY, g^{TY})$ . Observe also that if we denote by  $i_{y_0} : G \rightarrow Gy_0$  the map defined by  $i_{y_0}(g) = gy_0$ , then

$$(4.26) \quad \frac{1}{h^2(y)} dv_Y(y) = i_{y_0}^{-1*} dg,$$

this gives us a factor  $\frac{1}{h^2(y_0)}$  when we take the integral on  $\mathfrak{g}$  instead on the normal coordinate on  $X$ .

By (4.13), (4.15), (4.20), (4.24)-(4.26) and the Taylor expansion for  $\kappa_1$ ,  $f^E$ ,  $f^L$ , as in [1, Theorems 5.8, 5.9], we know that there exist  $J'_r(Z)$  polynomials in  $Z$  with same parity on  $r$ , and  $J'_0 = I_{\mathbb{C} \otimes E}$ , such that

$$(4.27) \quad P_p^G(y_0, y_0) \sim p^n \frac{1}{h^2(y_0)} \int_{K \in \mathfrak{g}, |K| \leq \varepsilon} e^{-\frac{\pi}{2}p|K|^2} \sum_{r=0}^{\infty} J'_r(\sqrt{p}K) p^{-r/2} dK.$$

Moreover,

$$(4.28) \quad \int_{K \in \mathfrak{g}} e^{-\frac{\pi}{2}p|K|^2} dK = 2^{\frac{n_0}{2}} p^{-\frac{n_0}{2}}.$$

After taking the integral on  $\mathfrak{g}$ , from (4.27) and (4.28), we get (0.15) and (0.16).

By (4.7), (4.27) and (4.28), we get also the asymptotic expansion for  $P_p^\vee(y_0, y_0)$ ,  $y_0 \in P$ .

**4.4. Toeplitz operators on  $X_G$ .** In this Subsection, we suppose that  $(X, \omega)$  is a Kähler manifold,  $\mathbf{J} = J$ , and  $L, E$  are holomorphic vector bundles with holomorphic Hermitian connections  $\nabla^L, \nabla^E$ . Let  $G$  be a compact connected Lie group acting holomorphically on  $X, L, E$  which preserves  $h^L$  and  $h^E$ .

We suppose that  $G$  acts freely on  $P = \mu^{-1}(0)$ . Then  $(X_G, \omega_G)$  is Kähler and  $L_G, E_G$  are holomorphic on  $X_G$ .

In this case, there exists a natural isomorphism from  $(\text{Ker } D_p)^G$  onto  $\text{Ker } D_{G,p}$ .

At the end of this Subsection, we will explain the corresponding result in the symplectic case, especially, for  $p \gg 1$ , we construct a natural isomorphism from  $(\text{Ker } D_p)^G$  onto  $\text{Ker } D_{G,p}$ .

In the current situation, the  $\text{spin}^c$  Dirac operator  $D_p$  was given by (0.21) and  $D_p^2$  preserves the  $\mathbb{Z}$ -grading of  $\Omega^{0,\bullet}(X, L^p \otimes E)$ . Similar properties hold for  $D_{G,p}$ .

As in Section 2.3, let  $P_{G,p}$  be the orthogonal projection from  $\Omega^{0,\bullet}(X_G, L_G^p \otimes E_G)$  onto  $\text{Ker } D_{G,p}$ , and let  $P_{G,p}(x, x')$  be the corresponding smooth kernel.

By the Kodaira vanishing theorem, for  $p$  large enough,

$$(4.29) \quad (\text{Ker } D_p)^G = H^0(X, L^p \otimes E)^G, \quad \text{Ker } D_{G,p} = H^0(X_G, L_G^p \otimes E_G).$$

As  $D_p^2, D_{G,p}^2$  preserve the  $\mathbb{Z}$ -gradings of  $\Omega^{0,\bullet}(X, L^p \otimes E), \Omega^{0,\bullet}(X_G, L_G^p \otimes E_G)$  respectively, we only need to take care of their restrictions on  $\mathcal{C}^\infty(X, L^p \otimes E)$  and  $\mathcal{C}^\infty(X_G, L_G^p \otimes E_G)$ . In this way,

$$(4.30) \quad \begin{aligned} P_p^G(x, x') &\in \mathcal{C}^\infty(X \times X, \text{pr}_1^*(L^p \otimes E) \otimes \text{pr}_2^*(L^p \otimes E)^*), \\ P_{G,p}(x_0, x'_0) &\in \mathcal{C}^\infty(X_G \times X_G, \text{pr}_1^*(L_G^p \otimes E_G) \otimes \text{pr}_2^*(L_G^p \otimes E_G)^*). \end{aligned}$$

Recall that the morphism  $\sigma_p : H^0(X, L^p \otimes E)^G \rightarrow H^0(X_G, L_G^p \otimes E_G)$  was defined in (0.27). Set

$$(4.31) \quad \sigma_p^G = \sigma_p \circ P_p^G : \mathcal{C}^\infty(X, L^p \otimes E) \rightarrow H^0(X_G, L_G^p \otimes E_G).$$

Let  $\sigma_p^{G*}$  be the adjoint of  $\sigma_p^G$  with respect to the natural inner products (cf. (1.19)) on  $\mathcal{C}^\infty(X, L^p \otimes E), \mathcal{C}^\infty(X_G, L_G^p \otimes E_G)$ . Set

$$(4.32) \quad \mathcal{P}_p^{X_G} := p^{-\frac{n_0}{2}} \sigma_p^G \circ \sigma_p^{G*}.$$

Let  $\{s_{p,i}\}_{i=1}^{d_p}$  be an orthonormal basis of  $H^0(X, L^p \otimes E)^G$ . For  $y_0 \in X_G$ ,  $x, x' \in X$ , one verifies

$$(4.33) \quad \begin{aligned} P_p^G(x, x') &= \sum_{i=1}^{d_p} s_{p,i}(x) \otimes s_{p,i}(x')^*, \\ \sigma_p^G(y_0, x) &= P_p^G(y_0, x), \quad \sigma_p^{G*}(x, y_0) = P_p^G(x, y_0), \end{aligned}$$

where by  $P_p^G(y_0, x)$  (resp.  $P_p^G(x, y_0)$ ) we mean  $P_p^G(y, x)$  (resp.  $P_p^G(x, y)$ ) for any  $y \in \pi_G^{-1}(y_0)$ , which is well-defined by the  $G$ -invariance of  $P_p^G$ .

From (0.27), we know that  $\mathcal{P}_p^{X_G}$  commutes with the operator  $P_{G,p}$  and

$$(4.34) \quad \mathcal{P}_p^{X_G} = P_{G,p} \mathcal{P}_p^{X_G} P_{G,p}.$$

Let  $P_p^G|_P$  be the restriction of the smooth kernel  $P_p^G(x, x')$  on  $P \times P$ . Then

$$P_p^G|_P(x, x') \in \mathcal{C}^\infty(P \times P, \text{pr}_1^*(L^p \otimes E) \otimes \text{pr}_2^*(L^p \otimes E)^*)$$

is  $G \times G$ -invariant. By composing with  $\pi_G$ ,

$$(\pi_G \circ P_p^G|_P)(x_0, x'_0) \in \mathcal{C}^\infty(X_G \times X_G, \text{pr}_1^*(L_G^p \otimes E_G) \otimes \text{pr}_2^*(L_G^p \otimes E_G)^*).$$

We denote by  $\pi_G \circ P_p^G|_P$  the operator defined by the smooth kernel  $(\pi_G \circ P_p^G|_P)(x_0, x'_0)$  and the volume form  $dv_{X_G}(x'_0)$ . Then from (4.33), we verify that

$$(4.35) \quad \mathcal{P}_p^{X_G}(x_0, x'_0) = p^{-\frac{n_0}{2}} P_p^G(x_0, x'_0) = p^{-\frac{n_0}{2}} \pi_G \circ P_p^G|_P(x_0, x'_0).$$

**Definition 4.3.** A family of operators  $T_p : H^0(X_G, L_G^p \otimes E_G) \rightarrow H^0(X_G, L_G^p \otimes E_G)$  is a Toeplitz operator if there exists a sequence of smooth sections  $g_l \in \mathcal{C}^\infty(X_G, \text{End}(E_G))$  with an asymptotic expansion  $g(\cdot, p)$  of the form  $\sum_{l=0}^\infty p^{-l} g_l(x)$  such that for any  $k \in \mathbb{N}^*$ , there exists  $C > 0$  such that for any  $p \in \mathbb{N}$ ,

$$(4.36) \quad \|T_p - P_{G,p} \sum_{l=0}^k p^{-l} g_l(x) P_{G,p}\|^{0,0} \leq C p^{-(k+1)}.$$

Here  $\|\cdot\|^{0,0}$  is the operator norm with respect to the norm  $\|\cdot\|_{L^2}$ . We call  $g_0(x)$  the principal symbol of  $T_p$ . If  $T_p$  is self-adjoint, then we call  $T_p$  is a self-adjoint Toeplitz operator.

Recall that  $h$  is the fiberwise volume function defined by (0.10).

Let  $dg$  be a Haar measure on  $G$ .

The main result of this Subsection is the following result.

**Theorem 4.4.** *Let  $f$  be a smooth section of  $\text{End}(E)$  on  $X$ . Let  $f^G \in \mathcal{C}^\infty(X_G, \text{End}(E_G))$  be the  $G$ -invariant part of  $f$  on  $P$  defined by  $f^G(x) = \int_G g f(g^{-1}x) dg$ . Then  $\mathcal{T}_{f,p} = p^{-\frac{n_0}{2}} \sigma_p^G f \sigma_p^{G*}$  is a Toeplitz operator with principal symbol  $2^{\frac{n_0}{2}} \frac{f^G}{h^2}(x)$ . In particular  $\mathcal{P}_p^{X_G}$  is a Toeplitz operator with principal symbol  $2^{\frac{n_0}{2}} / h^2(x)$ .*

*Proof.* Let  $f^*$  be the adjoint of  $f$ . By writing

$$f = \frac{f + f^*}{2} + \sqrt{-1} \frac{f - f^*}{2\sqrt{-1}},$$

we may and we will assume from now on that  $f$  is self-adjoint.

We need to find a family of sections  $g_l \in \mathcal{C}^\infty(X_G, \text{End}(E_G))$  such that for any  $m \geq 1$ ,

$$(4.37) \quad \mathcal{T}_{f,p} = \sum_{l=0}^m P_{G,p} g_l p^{-l} P_{G,p} + \mathcal{O}(p^{-m-1}).$$

Moreover, we can make these  $g_l$ 's to be self-adjoint.

Let  $U$  be a  $G$ -neighborhood of  $P = \mu^{-1}(0)$  as in Theorem 0.2.

Let  $\psi$  be a  $G$ -invariant function on  $X$  such that  $\psi = 1$  on a neighborhood of  $P$  and  $\text{supp}(\psi) \subset \{y \in X, d(y, P) < \varepsilon_0/2\} \cap U$ .

Write

$$(4.38) \quad \sigma_p^G f \sigma_p^{G*} = \sigma_p^G \psi f \sigma_p^{G*} + \sigma_p^G (1 - \psi) f \sigma_p^{G*}.$$

For  $x_0, x'_0 \in X_G$ , let  $x, x' \in P$  such that  $\pi(x) = x_0$ ,  $\pi(x') = x'_0$ . By (4.33),

$$(4.39) \quad (\sigma_p^G ((1 - \psi) f) \sigma_p^{G*})(x_0, x'_0) = \int_X P_p^G(x, y) ((1 - \psi) f)(y) P_p^G(y, x') dv_X(y).$$

From Theorem 0.1, (4.39) and  $\text{supp}((1 - \psi) f) \cap P = \emptyset$ , we know that for any  $l, m \in \mathbb{N}$ , there exists  $C_{l,m} > 0$  such that for any  $p \in \mathbb{N}$ ,  $x_0, x'_0 \in X_G$ ,

$$(4.40) \quad |(\sigma_p^G ((1 - \psi) f) \sigma_p^{G*})(x_0, x'_0)|_{\mathcal{C}^m(X_G \times X_G)} \leq C_{l,m} p^{-l}.$$

We define  $f_B \in \mathcal{C}^\infty(B, \text{End}(E_B))$  by

$$(4.41) \quad f_B(x_0) = \int_G g(\psi f)(g^{-1}x) dg$$

for  $x_0 \in B, x \in U$  such that  $\pi(x) = x_0$ . Clearly, if  $x_0 \in P$ , as  $\psi|_P = 1$ , one gets

$$(4.42) \quad f_B(x_0) = f^G(x_0).$$

From (4.41), for  $x_0, x'_0 \in B$ ,  $x, x' \in U$  such that  $\pi(x) = x_0$ ,  $\pi(x') = x'_0$ , one gets

$$(4.43) \quad \begin{aligned} \sigma_p^G \psi f \sigma_p^{G*}(x_0, x'_0) &= \int_U P_p^G(x, y) (\psi f)(y) P_p^G(y, x') dv_X(y) \\ &= \int_B P_p^G(x_0, y_0) f_B(y_0) P_p^G(y_0, x'_0) h^2(y_0) dv_B(y_0). \end{aligned}$$

For  $x_0 \in X_G$ , we will work on the normal coordinate of  $X_G$  with center  $x_0$  as in Theorem 0.2.

Recall that  $P_{\mathcal{L}}(Z^0, Z'^0)$  was defined by (3.19) with  $a_i = a_i^\perp = 2\pi$  therein.

By (4.39), (4.40) and (4.43), for  $|Z^0|, |Z'^0| \leq \varepsilon_0/2$ ,

$$(4.44) \quad \mathcal{T}_{f,p}(Z^0, Z'^0) - p^{-n_0/2} \int_{\substack{|W| \leq \varepsilon_0, \\ W \in T_{x_0} B}} P_p^G(Z^0, W) (f_B h^2)(W) P_p^G(W, Z'^0) dv_B(W) = \mathcal{O}(p^{-\infty}).$$

By Theorem 0.2, (4.44) and the Taylor expansion of  $f_B$ , there exist  $Q_{0,r} \in \text{End}(E_{G,x_0})$  polynomials on  $Z^0, Z'^0$  with same parity on  $r$  such that the following formula, obtained through compositions, holds,

$$(4.45) \quad \begin{aligned} &\left| p^{-n+n_0} \mathcal{T}_{f,p}(Z^0, Z'^0) - \sum_{r=0}^k (Q_{0,r} P_{\mathcal{L}})(\sqrt{p} Z^0, \sqrt{p} Z'^0) p^{-\frac{r}{2}} \right|_{\mathcal{C}^{m'}(X_G)} \\ &\leq C p^{-(k+1)/2} (1 + \sqrt{p}|Z^0| + \sqrt{p}|Z'^0|)^M \exp(-\sqrt{C''}\nu\sqrt{p}|Z^0 - Z'^0|) + \mathcal{O}(p^{-\infty}). \end{aligned}$$

On the normal coordinate in  $X_G$ , under the trivialization induced by the parallel transport of  $\nabla^{(L^p \otimes E)_G}$  along the geodesic, by [17, Theorem 4.18'] (i.e. Theorem 0.2 for  $G = 1$ ), we get : there exist  $J_r(Z^0, Z'^0) \in \text{End}(E_G)_{x_0}$ , polynomials in  $Z^0, Z'^0$  with the

same parity as  $r$ , such that for any  $k, m' \in \mathbb{N}$ , there exist  $M \in \mathbb{N}, C > 0$  such that for  $x_0 \in X_G$ ,  $Z^0, Z'^0 \in T_{x_0}X_G$ ,  $|Z^0|, |Z'^0| \leq \varepsilon$ ,

$$(4.46) \quad \left| p^{-n+n_0} P_{G,p}(Z^0, Z'^0) - \sum_{r=0}^k (J_r P_{\mathcal{L}})(\sqrt{p}Z^0, \sqrt{p}Z'^0) \kappa^{-\frac{1}{2}}(Z^0) \kappa^{-\frac{1}{2}}(Z'^0) p^{-\frac{r}{2}} \right|_{\mathcal{C}^{m'}(X_G)} \\ \leq C p^{-(k+1)/2} (1 + \sqrt{p}|Z^0| + \sqrt{p}|Z'^0|)^M \exp(-\sqrt{C''\nu} \sqrt{p}|Z^0 - Z'^0|) + \mathcal{O}(p^{-\infty}).$$

By using the Taylor expansion of  $\kappa^{-1/2}$ , from (4.46), there exist  $J_{0,r} \in \text{End}(E_{G,x_0})$ , polynomials on  $Z^0, Z'^0$  with same parity as  $r$ , such that

$$(4.47) \quad \left| p^{-n+n_0} P_{G,p}(Z^0, Z'^0) - \sum_{r=0}^k (J_{0,r} P_{\mathcal{L}})(\sqrt{p}Z^0, \sqrt{p}Z'^0) p^{-\frac{r}{2}} \right|_{\mathcal{C}^{m'}(X_G)} \\ \leq C p^{-(k+1)/2} (1 + \sqrt{p}|Z^0| + \sqrt{p}|Z'^0|)^M \exp(-\sqrt{C''\nu} \sqrt{p}|Z^0 - Z'^0|) + \mathcal{O}(p^{-\infty}).$$

Moreover, by (0.13) and (4.42), for  $Q_{0,0}, J_{0,0}$  in (4.45) and (4.47), we have

$$(4.48) \quad Q_{0,0} = 2^{\frac{n_0}{2}} \frac{f^G}{h^2}(x_0), \quad J_{0,0} = \text{Id}_{E_G}.$$

In what follows, all operators will be defined by their kernels with respect to  $dv_{T_{x_0}X_G}$ . We will add a subscript  $z^0$  or  $z'^0$  when we need to specify the operator acting on the variables  $Z^0$  or  $Z'^0$ .

By Theorem 3.1, we know that

$$(4.49) \quad b_{j,z^0}^+ P_{\mathcal{L}} = 0, \quad (b_j P_{\mathcal{L}})(Z^0, Z'^0) = b_{j,z^0} P_{\mathcal{L}}(Z^0, Z'^0) = 2\pi(\bar{z}_j^0 - \bar{z}_j'^0) P_{\mathcal{L}}(Z^0, Z'^0).$$

Thus for  $F(Z^0, Z'^0)$  a polynomial on  $Z^0, Z'^0$ , by (3.9), Theorem 3.1, (4.49), we can replace  $\bar{z}_j^0$  in  $F$  of  $(F P_{\mathcal{L}})(Z^0, Z'^0)$  by the combination of  $b_{j,z^0}$  and  $\bar{z}_j'^0$ , thus there exist polynomials  $F_{\alpha^0}$  ( $\alpha^0 \in \mathbb{N}^{n-n_0}$ ) on  $z^0, Z'^0$  (resp.  $F_{\alpha^0,0}$  on  $z^0, \bar{z}'^0$ ) such that

$$(4.50) \quad (F P_{\mathcal{L}})(Z^0, Z'^0) = \sum_{\alpha^0} b_{z^0}^{\alpha^0} (F_{\alpha^0} P_{\mathcal{L}})(Z^0, Z'^0), \\ ((F P_{\mathcal{L}}) \circ P_{\mathcal{L}})(Z^0, Z'^0) = \sum_{\alpha^0} b_{z^0}^{\alpha^0} F_{\alpha^0,0}(z^0, \bar{z}'^0) P_{\mathcal{L}}(Z^0, Z'^0).$$

In fact, by Theorem 3.1, the coefficient of  $P_{\mathcal{L}}(Z^0, Z'^0)$  in the right hand side of the second equation of (4.50) is anti-holomorphic on  $z'^0$ . Moreover, by Theorem 3.1,  $|\alpha^0| + \deg F_{\alpha^0}$ ,  $|\alpha^0| + \deg F_{\alpha^0,0}$  have the same parity with the degree of  $F$  on  $Z^0, Z'^0$ . In particular,  $F_{0,0}(z^0, \bar{z}'^0)$  is a polynomial on  $z^0, \bar{z}'^0$  and its degree has the same parity with  $\deg F$ .

We will denote by

$$(4.51) \quad F_{\mathcal{L}} := F_0, \quad F_{\mathcal{L},0} = F_{0,0}.$$

Let  $(F P_{\mathcal{L}})_p$  be the operator defined by the kernel  $p^{n-n_0} (F P_{\mathcal{L}})(\sqrt{p}Z^0, \sqrt{p}Z'^0)$ .

By Theorem 3.1, (4.47), (4.49), there exist polynomials  $H_r(F)$  on  $Z^0, Z'^0 \in T_{x_0}X_G$ , with the same parity with  $\deg F + r$ , such that we have the following asymptotic at



center  $x_0$ ,

$$(4.52) \quad P_{G,p}(FP_{\mathcal{L}})_p P_{G,p} \sim \sum_{r=0}^{\infty} (H_r(F)P_{\mathcal{L}})_p p^{-r/2}; \quad H_0(F) = F_{0,0} = F_{\mathcal{L},0},$$

with the reminder term estimated in the sense of (4.45) and (4.47).

By Theorem 3.1, (4.47), (4.50) and (4.52), the coefficient of  $p^{-k/2}$  in the expansion (4.52) of  $\sum_{r=0}^k P_{G,p}(Q_{0,r}P_{\mathcal{L}})_p p^{-\frac{r}{2}} P_{G,p}$  is

$$(4.53) \quad ((Q_{0,k})_{\mathcal{L},0}P_{\mathcal{L}})_p + \sum_{r=1}^k (H_r(Q_{0,k-r})P_{\mathcal{L}})_p.$$

Now, by (4.31),

$$(4.54) \quad \mathcal{T}_{f,p} = p^{-\frac{n_0}{2}} \sigma_p^G f \sigma_p^{G*} = P_{G,p} \mathcal{T}_{f,p} P_{G,p}.$$

Thus by (4.45), (4.53) and (4.54), we get

$$(4.55) \quad Q_{0,k} = (Q_{0,k})_{\mathcal{L},0} + \sum_{r=0}^{k-1} H_{k-r}(Q_{0,r}).$$

By (4.47) and (4.48), for  $\mathbf{f} \in \mathcal{C}^\infty(X_G, \text{End}(E_G))$ , there exist polynomials  $G_r(\mathbf{f})(Z^0, Z'^0)$  with the same parity as  $r$  such that in the normal coordinates as above,

$$(4.56) \quad P_{G,p} \mathbf{f} P_{G,p} \sim \sum_{r=0}^{\infty} (G_r(\mathbf{f})P_{\mathcal{L}})_p p^{-\frac{r}{2}}; \quad \text{with } G_0(\mathbf{f})(Z^0, Z'^0) = \mathbf{f}(x_0).$$

By (4.47), (4.56),  $(P_{G,p})^2 = P_{G,p}$  and by proceeding as in (4.55), we get

$$(4.57) \quad G_r(\mathbf{f}) = \sum_{i=0}^r H_i(G_{r-i}(\mathbf{f})) = (G_r(\mathbf{f}))_{\mathcal{L},0} + \sum_{i=1}^r H_i(G_{r-i}(\mathbf{f})).$$

From (4.48), we define

$$(4.58) \quad g_0(x) = 2^{\frac{n_0}{2}} \frac{f^G}{h^2}(x).$$

Assume that we have found  $g_l \in \mathcal{C}^\infty(X_G, \text{End}(E_G))$ , ( $l \leq k_0$ ), self-adjoint sections such that (4.37) holds for  $m = k_0$ .

We claim that  $Q_{0,2k_0+1}$  is determined by  $g_l$ , ( $l \leq k_0$ ), and there exists  $g_{k_0+1} \in \mathcal{C}^\infty(X_G, \text{End}(E_G))$  self-adjoint such that  $Q_{0,2k_0+2}$  is determined by  $g_l$  ( $l \leq k_0 + 1$ ).

By (4.37), (4.45) and (4.56), for  $0 < k \leq 2k_0$ ,

$$(4.59) \quad Q_{0,k} = \sum_{2l+j=k} G_j(g_l).$$

Then by (4.55), (4.57) and (4.59), for  $m = 2k_0 + 1$ ,

$$\begin{aligned}
 (4.60) \quad Q_{0,m} &= (Q_{0,m})_{\mathcal{L},0} + \sum_{r=0}^{m-1} H_{m-r} \left( \sum_{l=0}^{\lfloor r/2 \rfloor} G_{r-2l}(g_l) \right) \\
 &= (Q_{0,m})_{\mathcal{L},0} + \sum_{l=0}^{\lfloor \frac{m-1}{2} \rfloor} \sum_{r=2l}^{m-1} H_{m-r}(G_{r-2l}(g_l)) \\
 &= (Q_{0,m})_{\mathcal{L},0} + \sum_{l=0}^{\lfloor \frac{m-1}{2} \rfloor} \left( G_{m-2l}(g_l) - (G_{m-2l}(g_l))_{\mathcal{L},0} \right).
 \end{aligned}$$

Set

$$(4.61) \quad \mathcal{F}_m = (Q_{0,m})_{\mathcal{L},0} - \sum_{l=0}^{\lfloor \frac{m-1}{2} \rfloor} (G_{m-2l}(g_l))_{\mathcal{L},0}.$$

Then by (4.45), (4.52),  $\mathcal{F}_m$  is a polynomial on  $z^0, \bar{z}^0$  with the same parity as  $m$ . Moreover, as  $T_{f,p}$  and  $g_l$  are self-adjoint, we know that

$$(4.62) \quad \mathcal{F}_{m,x_0}^{(i)}(z^0, \bar{z}^0) = (\mathcal{F}_{m,x_0}^{(i)}(z'^0, \bar{z}'^0))^*.$$

Let  $\mathcal{F}_m^{(i)}$  be the degree  $i$  part of the polynomial  $\mathcal{F}_m$  on  $z^0, \bar{z}^0$ .

We need to prove that for  $m = 2k_0 + 1$ ,

$$(4.63) \quad \mathcal{F}_m^{(i)} = 0 \quad \text{for } i > 0.$$

Set

$$\begin{aligned}
 (4.64) \quad F_m^{(i)}(x_0, y_0) &= \mathcal{F}_{m,x_0}^{(i)}(0, \bar{z}^0) \in \text{End}(E_{G,x_0}), \\
 \tilde{F}_m^{(i)}(x_0, y_0) &= (F_m^{(i)}(y_0, x_0))^* \in \text{End}(E_{G,y_0}),
 \end{aligned}$$

with  $y_0 = \exp_{x_0}^{X_G}(Z'^0)$ , they define smooth sections on a neighborhood of the diagonal of  $X_G \times X_G$ . Clearly,  $\tilde{F}_m^{(i)}(x_0, y_0)$ 's need not be polynomials of  $z^0$  and  $\bar{z}^0$ .

Let  $\psi : \mathbb{R} \rightarrow [0, 1]$  be an even function such that  $\psi(u) = 1$  for  $|u| \leq \varepsilon_0/4$  and 0 for  $|u| > \varepsilon_0/2$ .

Let  $d^{X_0}(x_0, y_0)$  be the Riemannian distance on  $X_G$ .

We denote by  $(\psi F_m^{(i)} P_{G,p})$ ,  $(P_{G,p} \psi \tilde{F}_m^{(i)})$  the operators defined by the kernel  $(\psi(d^{X_0}) F_m^{(i)} P_{G,p})(x_0, y_0)$ ,  $(P_{G,p} \psi(d^{X_0}) \tilde{F}_m^{(i)})(x_0, y_0)$  with respect to  $dv_{X_G}(y_0)$ . Set

$$(4.65) \quad P_{p,k_0} = \mathcal{T}_{f,p} - P_{G,p} \sum_{l=0}^{k_0} g_l p^{-l} P_{G,p} - \sum_i (\psi F_{2k_0+1}^{(i)} P_{G,p}) p^{-(2k_0+i-1)/2}.$$

By (4.45), (4.56), (4.59) and (4.60),

$$\begin{aligned}
 (4.66) \quad & \left| p^{-n+n_0} \left( \mathcal{T}_{f,p} - P_{G,p} \sum_{l=0}^{k_0} g_l p^{-l} P_{G,p} \right) (0, Z'^0) - p^{-(2k_0+1)/2} (\mathcal{F}_{2k_0+1} P_{\mathcal{L}}) (0, \sqrt{p} Z'^0) \right| \\
 & \leq C p^{-k_0-1} (1 + \sqrt{p} |Z'^0|)^M \exp(-\sqrt{C''\nu} \sqrt{p} |Z'^0|) + \mathcal{O}(p^{-\infty}).
 \end{aligned}$$

Then by (4.53) and (4.66), there exist polynomials  $Q_{0,r,k_0}$  on  $Z^0, Z'^0$  with the same parity as  $r$  such that for  $k > 2k_0 + 2$ , the kernel of the operator  $P_{p,k_0}$  has the expansion at the normal coordinate of  $x_0$ , as

$$(4.67) \quad \left| p^{-n+n_0} P_{p,k_0}(0, Z'^0) - \sum_{r=2k_0+2}^k (Q_{0,r,k_0} P_{\mathcal{L}})(0, \sqrt{p} Z'^0) p^{-\frac{r}{2}} \right| \\ \leq C p^{-(k+1)/2} (1 + \sqrt{p} |Z'^0|)^M \exp(-\sqrt{C''\nu} \sqrt{p} |Z'^0|) + \mathcal{O}(p^{-\infty}).$$

We denote by  $Q_{0,r,k_0}^p$  the operator defined as in (4.65) by the kernel  $Q_{0,r,k_0}^p(x_0, y_0) = p^{n-n_0} \psi(d^{X_0}(x_0, y_0)) (Q_{0,r,k_0} P_{\mathcal{L}})(0, \sqrt{p} Z'^0)$ .

Set

$$(4.68) \quad K_{p,k}(x_0, y) = \psi(d^{X_0}(x_0, y)) P_{p,k_0}(x_0, y) - \sum_{r=2k_0+2}^k Q_{0,r,k_0}^p p^{-\frac{r}{2}}(x_0, y).$$

Then by (4.67),

$$(4.69) \quad |K_{p,k}(x_0, y)| \\ \leq C p^{n-n_0-(k+1)/2} (1 + \sqrt{p} d^{X_0}(x_0, y))^M \exp(-\sqrt{C''\nu} \sqrt{p} d^{X_0}(x_0, y)) + \mathcal{O}(p^{-\infty}).$$

Thus for any  $s \in \mathcal{C}^\infty(X_G, L_G^p \otimes E_G)$ ,

$$(4.70) \quad \|K_{p,k} s\|_{L^2}^2 \leq \int_{x_0 \in X_G} \left( \int_{y_0 \in X_G} |K_{p,k}(x_0, y_0)| dv_{X_G}(y_0) \right) \\ \times \left( \int_{y_0 \in X_G} |K_{p,k}(x_0, y_0)| |s|^2(y_0) dv_{X_G}(y_0) \right) dv_{X_G}(x_0) \\ \leq C p^{-(k+1)} \|s\|_{L^2}^2.$$

In the same way as in (4.70),

$$(4.71) \quad \|Q_{0,r,k_0}^p s\|_{L^2}^2 \leq C \|s\|_{L^2}^2.$$

Moreover, by Theorem 0.1, (4.65), we get

$$(4.72) \quad |((1 - \psi(d^{X_0})) P_{p,k_0})(x_0, y_0)| = \mathcal{O}(p^{-\infty}).$$

From (4.69), (4.70), (4.71) and (4.72), we know that there exists  $C > 0$  such that for any  $s \in \mathcal{C}^\infty(X_G, L_G^p \otimes E_G)$ ,

$$(4.73) \quad \|P_{p,k_0} s\|_{L^2} \leq C p^{-(k_0+1)} \|s\|_{L^2}.$$

Let  $P_{p,k_0}^*$  be the adjoint of  $P_{p,k_0}$ . By (4.73),

$$(4.74) \quad \|P_{p,k_0}^* s\|_{L^2} \leq C p^{-(k_0+1)} \|s\|_{L^2}.$$

But

$$(4.75) \quad P_{p,k_0}^* = \mathcal{T}_{f,p} - P_{G,p} \sum_{l=0}^{k_0} g_l p^{-l} P_{G,p} - \sum_i (P_{G,p} \psi \tilde{F}_{2k_0+1}^{(i)}) p^{(-2k_0+i-1)/2}.$$

By (4.47) and the Taylor expansion of  $\tilde{F}_{2k_0+1}^{(i)}$  under our trivialization of  $E_G$  by using parallel transport along the path  $[0, 1] \ni u \rightarrow uZ^0$ , we have that in the sense of (4.66),

$$(4.76) \quad p^{-n+n_0} \sum_i (P_{G,p} \tilde{F}_{2k_0+1}^{(i)})(0, Z^0) p^{(-2k_0+i-1)/2} \\ \sim \sum_{\alpha, i, r} (J_{0,r} P_{\mathcal{L}})(0, \sqrt{p}Z^0) \frac{\partial^\alpha \tilde{F}_{2k_0+1}^{(i)}(x_0, 0)}{\partial(Z^0)^\alpha} \frac{(\sqrt{p}Z^0)^\alpha}{\alpha!} p^{(-2k_0+i-|\alpha|-r-1)/2}.$$

By (4.66), (4.74), (4.75) and (4.76), we know that all coefficients of  $p^{(-2k_0-1+j)/2}$  for  $j > 0$  of the right hand side of (4.76) should be zero. Thus we get for any  $j > 0$ ,

$$(4.77) \quad \sum_{i=j}^{\deg F_m} \sum_{|\alpha|+r=0}^{i-j} J_{0,r}(0, \sqrt{p}Z^0) \frac{\partial^\alpha \tilde{F}_{2k_0+1}^{(i)}(x_0, 0)}{\partial(Z^0)^\alpha} \frac{(\sqrt{p}Z^0)^\alpha}{\alpha!} = 0.$$

From (4.77), we will prove by recurrence that for any  $j > 0$

$$(4.78) \quad \frac{\partial^\alpha \tilde{F}_{2k_0+1}^{(i)}(x_0, 0)}{\partial(Z^0)^\alpha} = 0 \quad \text{for } i - |\alpha| \geq j > 0.$$

In fact, for  $j = \deg F_{2k_0+1}$  in (4.77), by (4.48), we get  $\tilde{F}_{2k_0+1}^{(\deg F_{2k_0+1})}(x_0, 0) = 0$ , thus (4.78) holds for  $j = \deg F_{2k_0+1}$ .

Assume that for  $j > j_0 > 0$ , (4.78) holds. Then for  $j = j_0$ , the coefficient with  $r > 0$  in (4.77) is zero, thus by (4.48), (4.77) reads as

$$(4.79) \quad \sum_{\alpha} \frac{\partial^\alpha \tilde{F}_{2k_0+1}^{(j_0+|\alpha|)}(x_0, 0)}{\partial(Z^0)^\alpha} \frac{(\sqrt{p}Z^0)^\alpha}{\alpha!} = 0.$$

From (4.79), we get (4.78) for  $j = j_0$ . The proof of (4.78) is complete.

By (4.74), (4.76) and (4.78), by comparing the coefficient of  $p^{-(2k_0+1)/2}$  in (4.66) and (4.75), we get

$$(4.80) \quad \tilde{F}_{2k_0+1}^{(i)}(x_0, Z^0) = \mathcal{F}_{2k_0+1, x_0}^{(i)}(0, \bar{z}^0) + \mathcal{O}(|Z^0|^{i+1}).$$

Thus from (4.64) and (4.80)

$$(4.81) \quad F_{2k_0+1}^{(i)}(Z^0, x_0) = (\mathcal{F}_{2k_0+1, x_0}^{(i)}(0, \bar{z}^0))^* + \mathcal{O}(|Z^0|^{i+1}).$$

Let  $j_{X_G} : X_G \rightarrow X_G \times X_G$  be the diagonal injection. By (4.64),

$$(4.82) \quad \frac{\partial}{\partial z_j^0} F_{2k_0+1}^{(i)} = 0 \quad \text{near } j_{X_G}(X_G).$$

By (4.64) again and recurrence, for  $\alpha \in \mathbb{N}^{n-n_0}$ , if  $\alpha_j > 0$ , by taking  $\alpha' = (\alpha_0, \dots, \alpha_j - 1, \dots, \alpha_{n-n_0})$ , one has

$$(4.83) \quad \frac{\partial^\alpha}{\partial z_{0,\alpha}} F_{2k_0+1}^{(i)}(\cdot, x_0)|_{x_0} = \frac{\partial}{\partial z_j^0} j_{X_G}^* \left( \frac{\partial^{\alpha'}}{\partial z_{0,\alpha'}} F_{2k_0+1}^{(i)} \right) \Big|_{x_0} - \frac{\partial^{\alpha'}}{\partial z_{0,\alpha'}} \frac{\partial}{\partial z_j^0} F_{2k_0+1}^{(i)}(\cdot, \cdot) \Big|_{0,0} = 0.$$

But by (4.81), for  $|\alpha| \leq i$ ,

$$(4.84) \quad \frac{\partial^\alpha}{\partial z_{0,\alpha}} F_{2k_0+1}^{(i)}(\cdot, x_0)|_{x_0} = \left( \left( \frac{\partial^\alpha}{\partial \bar{z}^0, \alpha} \mathcal{F}_{2k_0+1, x_0}^{(i)} \right) (0, \bar{z}^0) \right)^*.$$

From (4.83) and (4.84), the  $\alpha$ -derivative for  $|\alpha| \leq i$  of  $F_{m,x_0}^{(i)}(x_0, \cdot)$  is zero at  $x_0$ . Thus

$$(4.85) \quad \mathcal{F}_{m,x_0}^{(i)}(0, \bar{z}^0) = \mathcal{F}_{m,x_0}^{(i)}(z^0, 0) = 0.$$

Now, we consider the operator

$$(4.86) \quad \frac{1}{\sqrt{p}} P_{G,p} \left( \nabla_{\varphi(d^{X_0}(x_0, y_0))(\frac{\partial}{\partial z_j^0} + \frac{\partial}{\partial \bar{z}_j^0})_{x_0}}^{L^p \otimes E} \left( \mathcal{T}_{f,p} - \sum_{l=0}^{k_0} P_{G,p} g_l p^{-l} P_{G,p} \right) \right) P_{G,p},$$

then the leading term of its asymptotic expansion as in (4.45) is

$$(4.87) \quad \left( \frac{\partial}{\partial z_j^0} \mathcal{F}_{2k_0+1,x_0} \right) (\sqrt{p} z^0, \sqrt{p} \bar{z}^0) P_{\mathcal{L}} (\sqrt{p} z^0, \sqrt{p} \bar{z}^0) p^{-(2k_0+1)/2}.$$

Here by (4.85),  $(\frac{\partial}{\partial z_j^0} \mathcal{F}_{2k_0+1,x_0})(z^0, \bar{z}^0)$  is an even degree polynomial on  $z^0, \bar{z}^0$ , and its constant term is zero. Now, by proceeding as (4.65)–(4.84) for the operator (4.86), by (4.62), we get

$$(4.88) \quad \frac{\partial}{\partial z_j^0} \mathcal{F}_{2k_0+1,x_0}^{(i)}(z^0, \bar{z}^0) = \frac{\partial}{\partial \bar{z}_j^0} \mathcal{F}_{2k_0+1,x_0}^{(i)}(z^0, \bar{z}^0) = 0.$$

By continuous this process, we get (4.63).

This means that  $Q_{0,2k_0+1}$  verifies also (4.59).

By the same argument, (4.60) still holds for  $m = 2k_0 + 2$ . Thus we can define

$$(4.89) \quad g_{k_0+1}(x_0) = \mathcal{F}_{2k_0+2,x_0}^{(0)} = \mathcal{F}_{2k_0+2,x_0}^{(0)}(0, 0).$$

By proceeding exactly the same proof as before, we get (4.63) for  $m = 2k_0 + 2$ . Thus for  $k = 2k_0 + 2$ , (4.59) still holds.

As  $\pi_{G,p}, \mathcal{T}_{f,p}, g_l$  ( $1 \leq l \leq k_0$ ) are self-adjoint,  $g_{k_0+1}$  is also self-adjoint.

By recurrence, we know that there exist  $g_l$ 's such that (4.37) holds for any  $m$ .

The proof of Theorem 4.4 is complete.  $\square$

**Corollary 4.5.** *For  $f_1, f_2 \in \mathcal{C}^\infty(X)$ , we have*

$$(4.90) \quad [\mathcal{T}_{f_1,p}, \mathcal{T}_{f_2,p}] = \frac{2^{n_0}}{\sqrt{-1}p} P_{G,p} \left\{ \frac{f_1^G}{h^2}, \frac{f_2^G}{h^2} \right\} P_{G,p} + \mathcal{O}(p^{-2}).$$

Here  $\{, \}$  is the Poisson bracket on  $(X_G, 2\pi\omega_G)$ : for  $g_1, g_2 \in \mathcal{C}^\infty(X_G)$ , if  $\xi_{g_2}$  is the Hamiltonian vector field generated by  $g_2$  which is defined by  $2\pi i_{\xi_{g_2}} \omega_G = dg_2$ , then

$$(4.91) \quad \{g_1, g_2\} = \xi_{g_2}(dg_1).$$

*Proof.* By applying [27] or [28, §5.5], (cf. [9] for another approach where they worked for  $E = \mathbb{C}$ ), from Theorem 4.4, we get immediately (4.90).  $\square$

**Lemma 4.6.** *Let*

$$T_p = \sum_{l=0}^{\infty} P_{G,p} g_l p^{-l} P_{G,p} + \mathcal{O}(p^{-\infty}) : H^0(X_G, L_G^p \otimes E_G) \rightarrow H^0(X_G, L_G^p \otimes E_G)$$

be a Toeplitz operator with principal symbol  $g_0 \in \mathcal{C}^\infty(X_G, \text{End}(E_G))$ . Then

i) If  $g_0$  is invertible, then  $T_p^{-1}$  is a Toeplitz operator with principal symbol  $g^{-1} \text{Id}_{E_G}$ .

ii) If  $g_0 = g \text{Id}_{E_G}$  with  $g \in \mathcal{C}^\infty(X_G)$ ,  $g > 0$ , and  $T_p$  is self-adjoint, then  $T_p^{1/2}$  is a self-adjoint Toeplitz operator with principal symbol  $g^{1/2} \text{Id}_{E_G}$ .

*Proof.* We only prove ii), the proof of i) is similar and simpler.

As  $g > 0$ , there exist  $C_0, C_1 > 0$  such that  $C_0 < g < C_1$ . Thus for any  $s \in H^0(X_G, L_G^p \otimes E_G)$ ,

$$(4.92) \quad \langle T_p s, s \rangle = \langle g_0 s, s \rangle + \mathcal{O}\left(\frac{1}{p}\right) \|s\|_{L^2}^2 \geq \left(C_0 + \mathcal{O}\left(\frac{1}{p}\right)\right) \|s\|_{L^2}^2.$$

Thus for  $p$  large enough,  $T_p^{1/2} : H^0(X_G, L_G^p \otimes E_G) \rightarrow H^0(X_G, L_G^p \otimes E_G)$  is well defined.

Let  $\delta_1$  be the smooth bounded closed contour on  $\{\lambda \in \mathbb{C}, \text{Re}(\lambda) > 0\}$  such that  $[\frac{1}{2}C_0, 2C_1]$  is in the interior domain got by  $\delta_1$ .

As in the proof of Theorem 4.4, by recurrence, we will find  $f_l \in \mathcal{C}^\infty(X_G, \text{End}(E_G))$  such that

$$(4.93) \quad T_p = (T_{m,p})^2 + \mathcal{O}(p^{-m-1}) \quad \text{with} \quad T_{m,p} = \sum_{l=0}^m P_{G,p} f_l p^{-l} P_{G,p}.$$

Then for  $p$  large enough,

$$(4.94) \quad \begin{aligned} T_p^{1/2} - T_{m,p} &= \frac{1}{2\pi i} \int_{\lambda \in \delta_1} \lambda^{1/2} [(\lambda - T_p)^{-1} - (\lambda - (T_{m,p})^2)^{-1}] d\lambda \\ &= \frac{1}{2\pi i} \int_{\lambda \in \delta_1} \lambda^{1/2} (\lambda - T_p)^{-1} (T_p - (T_{m,p})^2) (\lambda - (T_{m,p})^2)^{-1} d\lambda. \end{aligned}$$

If (4.93) holds, then by (4.94) we know that in the sense of the operator norm,

$$(4.95) \quad T_p^{1/2} - T_{m,p} = \mathcal{O}(p^{-m-1}).$$

To complete the proof of Lemma 4.6, it remains to establish (4.93).

By (4.47), there exist  $Q_{0,r} \in \text{End}(E_G)_{x_0}$  such that in the sense of (4.45), (4.47) and (4.52),

$$(4.96) \quad T_p \sim \sum_{r=0}^{\infty} (Q_{0,r} P_{\mathcal{L}})_p p^{-r/2}.$$

We will prove by recurrence that there exist  $f_l \in \mathcal{C}^\infty(X_G, \text{End}(E_G))$  self-adjoint such that for any  $k \in \mathbb{N}$ ,

$$(4.97) \quad \begin{aligned} &\left| p^{-n+n_0} (T_p - (T_{k,p})^2) (\sqrt{p} Z^0, \sqrt{p} Z^0) \right| \\ &\leq p^{-(2k+1)/2} (1 + \sqrt{p} |Z^0| + \sqrt{p} |Z^0|)^M \exp(-\sqrt{C''} \nu \sqrt{p} |Z^0 - Z^0|) + \mathcal{O}(p^{-\infty}). \end{aligned}$$

Set  $f_0 = g^{1/2} \text{Id}_{E_G}$ . Then (4.93) is verified for  $m = 0$ .

Assume that for  $k \leq m$ , we have found  $f_l$  such that (4.93) holds. If we denote the expansion of  $(T_{m,p})^2$  in the sense of (4.47),

$$(4.98) \quad (T_{m,p})^2 \sim \sum_{r=0}^{\infty} (\tilde{Q}_{0,r}^m P_{\mathcal{L}})_p p^{-r/2}.$$

Then by the proof of (4.59) for  $2k_0 + 1$ ,

$$(4.99) \quad \tilde{Q}_{0,2m+1}^m = \sum_{2l+r=2m+1} G_r(g_l) = Q_{0,2m+1}.$$

Thus by (4.99), (4.97) stills holds when we replace the factor  $p^{-(2m+1)/2}$  by  $p^{-m-1}$  at the right hand side of (4.97). Thus

$$(4.100) \quad T_p - (T_{k,p})^2 \sim \sum_{r=2m+2}^{\infty} ((Q_{0,r} - \tilde{Q}_{0,r}^m)P_{\mathcal{L}})_p p^{-r/2}.$$

By (4.52), (4.100), we know that

$$(4.101) \quad Q_{0,2m+2} - \tilde{Q}_{0,2m+2}^m = (Q_{0,2m+2} - \tilde{Q}_{0,2m+2}^m)_{\mathcal{L},0}.$$

This means that  $Q_{0,2m+2} - \tilde{Q}_{0,2m+2}^m$  is a polynomial on  $z^0, \bar{z}^0$  with even degree.

Set

$$(4.102) \quad f_{m+1}(x_0) = -\frac{1}{2}g^{-1/2}(Q_{0,2m+2} - \tilde{Q}_{0,2m+2}^m)(0,0).$$

Then by the proof of (4.59), for  $2k_0 + 2$ , we know that the polynomial  $Q_{0,2m+2} - \tilde{Q}_{0,2m+2}^m$  equals to the constant  $-2g^{1/2}f_{m+1}$ . Thus we prove (4.97) for  $k = m + 1$ .

By the above argument, we have established (4.93), thus Lemma 4.6.  $\square$

Since the isomorphism  $\sigma_p : H^0(X, L^p \otimes E)^G \rightarrow H^0(X_G, L_G^p \otimes E_G)$  is not an isometry, we define the associated unitary operator,

$$(4.103) \quad \Sigma_p = \sigma_p^{G*}(\sigma_p^G \circ \sigma_p^{G*})^{-1/2} : H^0(X_G, L_G^p \otimes E_G) \rightarrow H^0(X, L^p \otimes E)^G.$$

**Theorem 4.7.** *Let  $f$  be a  $\mathcal{C}^\infty$  section of  $\text{End}(E)$  on  $X$ . Then*

$$(4.104) \quad T_{f,p}^G = \Sigma_p^* f \Sigma_p : H^0(X_G, L_G^p \otimes E_G) \rightarrow H^0(X_G, L_G^p \otimes E_G)$$

*is a Toeplitz operator on  $X_G$ . Its principal symbol is  $f^G \in \mathcal{C}^\infty(X_G, \text{End}(E_G))$ .*

*Proof.* By (4.34) and (4.103),

$$(4.105) \quad T_{f,p}^G = (\mathcal{P}_p^{X_G})^{-\frac{1}{2}} \mathcal{T}_{f,p} (\mathcal{P}_p^{X_G})^{-\frac{1}{2}}.$$

By Theorem 4.4, (4.32),  $\mathcal{P}_p^{X_G} = p^{-\frac{n_0}{2}} \sigma_p^G \circ \sigma_p^{G*}$ ,  $\mathcal{T}_{f,p}$  are Toeplitz operators on  $X_G$  with principal symbols  $2^{n_0/2}/h^2(x)$ ,  $2^{n_0/2} \frac{f^G}{h^2}(x)$  respectively.

By Lemma 4.6, we know that  $(\mathcal{P}_p^{X_G})^{-\frac{1}{2}}$  is a Toeplitz operator on  $X_G$ .

By (4.105),  $T_{f,p}^G$  has the expansion as (4.45). By the proof of Theorem 4.4, we then know that  $T_{f,p}^G$  is a Toeplitz operator.  $\square$

**Remark 4.8.** i) Certainly, by combining the argument here and Section 4.1, we can get the corresponding version when  $X_G$  is an orbifold.

ii) When  $E = \mathbb{C}$ , and  $f = 1$ , from Theorem 4.4,  $\mathcal{P}_p^{X_G}$  is an elliptic (i.e. its principal symbol is invertible) Toeplitz operator. This is the analytic core result claimed in [34, §8].

iii) When  $E = \mathbb{C}$  and  $G$  is the torus  $\mathbb{T}^{n_0}$ , Theorem 4.7 is one of the main results of Charles [15, Theorem 1.2], and in [15, §5.6], he knew also that  $\mathcal{P}_p^{X_G}$  is an elliptic Toeplitz operator. Moreover, he established the corresponding version when  $X_G$  is an orbifold.

If  $X$  is only symplectic and  $\mathbf{J} = J$ , then as the argument in [40, §3e)],  $J$  induces an almost complex structure  $J_G$  on  $(TX)_B$ , and  $J_G$  preserves  $N_{G,J} = N_G \oplus J_G N_G$  and  $TX_G$ . Thus one can construct canonically the Hermitian vector bundles  $N_{G,J}^{(1,0)}$  etc, which further gives the canonical identification of Hermitian vector bundles

$$(4.106) \quad \Lambda(T^{*(0,1)}X)_B|_{X_G} = \Lambda(N_{G,J}^{*(0,1)}) \widehat{\otimes} \Lambda(T^{*(0,1)}X_G).$$

Let  $q$  be the canonical orthogonal projection

$$(4.107) \quad q : \Lambda(N_{G,J}^{*(0,1)}) \widehat{\otimes} \Lambda(T^{*(0,1)}X_G) \otimes L_G^p \otimes E_G \rightarrow \Lambda(T^{*(0,1)}X_G) \otimes L_G^p \otimes E_G$$

which acts as identity on  $\Lambda(T^{*(0,1)}X_G) \otimes L_G^p \otimes E_G$  and maps each  $\Lambda^i(N_{G,J}^{*(0,1)}) \widehat{\otimes} \Lambda(T^{*(0,1)}X_G) \otimes L_G^p \otimes E_G$ ,  $i \geq 1$ , to zero.

We define

$$(4.108) \quad \sigma_p := P_{G,p} q \pi_G i^* P_p^G : (\text{Ker } D_p)^G \rightarrow \text{Ker } D_{G,p}.$$

Certainly in the Kähler case,  $\sigma_p$  coincides with (0.27).

By using Theorems 0.1, 0.2 as in the proof of Theorem 4.4 (cf. [27], [28, §5.5] for more details on the Toeplitz operators in the symplectic setting), we get

**Theorem 4.9.** *Let  $f$  be a smooth section of  $\text{End}(E)$  on  $X$ , then  $\mathcal{T}_{f,p} = \sigma_p f \sigma_p^* : \text{Ker } D_{G,p} \rightarrow \text{Ker } D_{G,p}$  is a Toeplitz operator with principal symbol  $2^{n_0/2} \frac{f_G}{h^2}(x) I_{\mathbb{C} \otimes E_G} \in \text{End}(\Lambda(T^{*(0,1)}X_G) \otimes E_G)$ .*

**Corollary 4.10.** *For  $p$  large enough,  $\sigma_p$  in (4.108) is an isomorphism. Thus  $\sigma_p$  defines a natural identification for ‘quantization commutes with reduction’ in the (asymptotic) symplectic case.*

*Proof.* From Theorem 4.9 for  $f = 1$ , we get

$$(4.109) \quad \sigma_p \sigma_p^* = 2^{n_0/2} P_{G,p} h^{-2} I_{\mathbb{C} \otimes E_G} P_{G,p} + \mathcal{O}\left(\frac{1}{p}\right).$$

But from the argument as (4.71) and Theorem 0.2 for  $G = 1$ , we get for any  $s \in \Omega^{0,\bullet}(X_G, L_G^p \otimes E_G)$ , we have

$$(4.110) \quad \|(I_{\mathbb{C} \otimes E_G} P_{G,p} - P_{G,p})s\|_{L^2} \leq \frac{C}{\sqrt{p}} \|s\|_{L^2}.$$

Thus for  $p$  large enough,  $\sigma_p \sigma_p^*$  is an isomorphism. Thus  $\sigma_p$  is surjective.

In view of (0.6),  $\sigma_p$  in (4.108) is an isomorphism.  $\square$

**Remark 4.11.** If we replace the condition  $\mathbf{J} = J$  by (3.2), then the canonical map  $\sigma_p$  in (4.108) is still well defined. From the argument here, we still know that  $\sigma_p$  is an isomorphism for  $p$  large enough.



**4.5. Generalization to non-compact manifolds.** In this Subsection, let  $(X, \omega)$  be a symplectic manifold, and  $(L, \nabla^L)$  (resp.  $(E, \nabla^E)$ ) be Hermitian line (vector) bundle on  $X$ , and the compact connected Lie group  $G$  acts on  $X$  as in Introduction, especially,  $\omega = \frac{\sqrt{-1}}{2\pi} R^L$ . But we only suppose that  $(X, g^{TX})$  is a complete manifold.

If  $G = 1$ , these kind results were studied in [26, §3].

By the argument in Section 2.3, if the square of the  $\text{spin}^c$  Dirac operator  $D_p^2$  has a spectral gap as in (2.13), then we can localize our problem and get a version of Theorems 0.1, 0.2 from Section 2.6. In particular, if the geometric data on  $X$  verify the bounded geometry, then  $D_p^2$  verify the spectral gap (2.13).

We explain in more detail now.

We suppose

- i) The tensors  $R^E, r^X, \text{Tr}[R^{T^{(1,0)}X}]$  are uniformly bounded with respect on  $(X, g^{TX})$ .
- ii) There exists  $c > 0$  such that

$$(4.111) \quad \sqrt{-1}R^L(\cdot, J\cdot) \geq cg^{TX}(\cdot, \cdot).$$

**Remark 4.12.** For the operator  $D_p = \sqrt{2}(\bar{\partial}^{L^p \otimes E} + \bar{\partial}^{L^p \otimes E, *})$  in the holomorphic case, the above condition i) can be replaced by

- i') The tensors  $R^E, R^{T^{(1,0)}X}, \partial T$  is uniformly bounded with respect to  $(X, g^{TX})$ , here  $T$  is the torsion of  $(X, \omega)$  as in [26, §3.5].

Then by the argument in [25, p. 656] (cf. [26, §3]), we know that Theorem 2.2 still holds. Thus Theorem 2.5 still holds.

Let  $P_p^G$  be the orthogonal projection from  $L^2(X, E_p)$  onto  $(\text{Ker } D_p)^G$ , and  $P_p^G(x, x')$  ( $x, x' \in X$ ) be its kernel as in Def. 2.3.

Note that  $\text{Ker } D_p$  and  $(\text{Ker } D_p)^G$  need not be finite dimensional.

By the proof of Prop. 2.6, we know that for any compact set  $K \subset X$ ,  $l, m \in \mathbb{N}$ , there exist  $C_{l,m}(K) > 0$  such that for  $p \geq C_L/\nu$ ,

$$(4.112) \quad |\tilde{F}(\mathcal{L}_p)(x, x') - P_p^G(x, x')|_{\mathcal{C}^m(K \times K)} \leq C_{l,m}(K)p^{-l}.$$

By the proof of Theorem 0.1, we get

**Theorem 4.13.** *For any compact set  $K \subset X$ ,  $0 < \varepsilon_0 \leq \delta_0$ ,  $l, m \in \mathbb{N}$ , there exists  $C_{l,m} > 0$  (depend on  $K, \varepsilon$ ) such that for  $p \geq 1$ ,  $x, x' \in K$ ,  $d^X(Gx, x') \geq \varepsilon_0$  or  $x, x' \in (X \setminus X_{2\varepsilon_0}) \cap K$ ,*

$$(4.113) \quad |P_p^G(x, x')|_{\mathcal{C}^m} \leq C_{l,m}p^{-l}.$$

From Section 2.6, we get Theorem 0.2, but now the norm  $\mathcal{C}^{m'}(X_G)$  in (0.14) should be replaced by  $\mathcal{C}^{m'}(K)$  for any compact set  $K \subset X_G$ .

One interesting case of the above discussion is when  $P = \mu^{-1}(0)$  is compact, by the same argument as in Theorems 4.4, 4.9, we can prove a version of Section 4.4.

In fact, when  $X = \mathbb{C}^n$ ,  $G = \mathbb{T}^{n_0}$ ,  $L$  is the trivial line bundle with the metric  $|1|_{h^L}(Z) = e^{-|z|^2}$ , the Toeplitz operator type properties was studied by Charles [15].

Another interesting case is a version of Theorem 0.2 for covering manifolds.

Let  $\tilde{X}$  be a para-compact smooth manifold, such that there is a discrete group  $\Gamma$  acting freely on  $\tilde{X}$  with a compact quotient  $X = \tilde{X}/\Gamma$ .

Let  $\pi_\Gamma : \tilde{X} \rightarrow X$  be the projection. Assume that all above geometric data on  $X$  can be lift on  $\tilde{X}$ . We denote by  $\tilde{\mathbf{J}}, g^{T\tilde{X}}, \tilde{\omega}, \tilde{J}, \tilde{L}, \tilde{E}$  the pull-back of the corresponding objects in Section 0 by the projection  $\pi_\Gamma : \tilde{X} \rightarrow X$ , moreover, we assume that the  $G$ -action and the  $\Gamma$ -action on them commute.

By the above arguments (cf. [25, Theorems 4.4 and 4.6]), there exists a spectral gap for the square of the spin<sup>c</sup> Dirac operator  $\tilde{D}_p$  on  $\tilde{X}$ .

By the finite propagation speed of solutions of hyperbolic equations (2.75), we get an extension of [26, Theorem 3.13] where  $G = 1$ .

**Theorem 4.14.** *We fix  $0 < \varepsilon_0 < \inf_{x \in X} \{\text{injectivity radius of } x\}$ . For any compact set  $K \subset \tilde{X}$  and  $k, l \in \mathbb{N}$ , there exists  $C_{k,l,K} > 0$  such that for  $x, x' \in K$ ,  $p \in \mathbb{N}$ ,*

$$(4.114) \quad \begin{aligned} & \left| \tilde{P}_p^G(x, x') - P_p^G(\pi_\Gamma(x), \pi_\Gamma(x')) \right|_{\mathcal{C}^l(K \times K)} \leq C_{k,l,K} p^{-k-1}, \quad \text{if } d^X(x, x') < \varepsilon_0, \\ & \left| \tilde{P}_p^G(x, x') \right|_{\mathcal{C}^l(K \times K)} \leq C_{k,l,K} p^{-k-1}, \quad \text{if } d^X(x, x') \geq \varepsilon_0. \end{aligned}$$

*Epecially,  $\tilde{P}_p^G(x, x)$  has the same asymptotic expansion as  $P_p^G(\pi_\Gamma(x), \pi_\Gamma(x))$  in Corollary 0.4 on any compact set  $K \subset \tilde{X}$ .*

**4.6. Relation on the Bergman kernel on  $X_G$ .** From (2.60), if the operator  $\Phi \mathcal{L}_p \Phi^{-1}$  has the form  $D_{G,p}^2 + \Delta_N + 4\pi|\mu|^2 - 2\pi n_0$  under the splitting (4.106), then we will find the full asymptotic expansion of the Bergman kernel on  $X_G$  from  $P_p^G(x, x')$ .

In this Subsection, we suppose that  $X$  is compact and  $G$  is a torus  $\mathbb{T}^{n_0} = \mathbb{R}^{n_0}/\mathbb{Z}^{n_0}$ .

Let  $\theta : TP \rightarrow \mathfrak{g}$  be a connection form for the  $G$ -principal bundle  $\pi : P = \mu^{-1}(0) \rightarrow X_G$  with curvature  $\Theta$ . Let  $T^H P = \text{Ker } \theta \subset TP$ .

Set  $M = P \times \mathfrak{g}^*$ ,  $\mathbf{q} : M \rightarrow \mathfrak{g}^*$  be the natural projection and

$$(4.115) \quad \omega^M = \pi^* \omega_G + d\langle \mathbf{q}, \theta \rangle = \pi^* \omega_G + \langle \mathbf{q}, \Theta \rangle + \langle d\mathbf{q}, \theta \rangle.$$

By the normal crossing formula [21, Prop. 40.1], we know there exists a symplectic diffeomorphism such that on a neighborhood  $U$  of  $P$ ,

$$(4.116) \quad \Psi_{sym} : (X, \omega) \simeq (M, \omega^M),$$

and under this identification, the moment map  $\mu$  (cf. (2.14)) is defined by  $-\mathbf{q}$ .

From now on, we use this neighborhood of  $P$  and we will choose metrics and connections.

Let  $g^{\mathfrak{g}}$  be the metric on  $\mathfrak{g}$  induced by the canonical flat metric on  $\mathbb{R}^{n_0}$ , and  $\{K_i\}$  be the canonical unitary basis of  $\mathbb{R}^{n_0}$ .

Now we choose  $J$  an almost-complex structure on  $TX$  compatible with  $\omega$  such that on  $T^H P$  on  $U$ ,  $J$  is induced by an almost-complex structure on  $TX_G$  which is compatible with  $\omega_G$ , and on  $\mathfrak{g} \oplus \mathfrak{g}^*$ , for  $K \in \mathfrak{g}$ ,  $JK \in \mathfrak{g}^*$  is defined by  $(JK, K') = \langle K, K' \rangle_{\mathfrak{g}}$  for  $K' \in \mathfrak{g}$ .

We also suppose  $\Theta$  is  $J$ -invariant.

Let  $g^{TX}$  be a  $J$ -invariant metric on  $TX$  such that

$$(4.117) \quad g^{TX} = \pi^* g^{TX_G} \oplus g^{\mathfrak{g}} \oplus g^{\mathfrak{g}^*} \quad \text{on } U.$$

As  $g^{\mathfrak{g}}$  is a constant metric on  $TY = \mathfrak{g}$ ,  $\nabla^{TY}$  is the trivial connection on  $TY$ . By (1.7), on  $U$ ,

$$(4.118) \quad \nabla_{U_1^H}^{TP} = \nabla_{U_1^H}^{TX_G} + \nabla_{U_1^H}^{TY} + S(U_1^H).$$

Let  $\nabla^{\Lambda(N_{G,J}^{*(0,1)})}$  be the trivial connection on the trivial bundle  $\Lambda(N_{G,J}^{*(0,1)})$  (cf. (4.106)) on  $U$ , and  $\nabla^{\text{Cliff}_{X_G}}$  be the Clifford connection on  $\Lambda(T^{*(0,1)}X_G)$ .

By (4.118), under the identification (4.106), on  $U$ , we have

$$(4.119) \quad \begin{aligned} \nabla_{e_i^H}^{\text{Cliff}} &= \nabla_{e_i^H}^{\text{Cliff}_{X_G}} \otimes \text{Id} + \text{Id} \otimes \nabla_{e_i^H}^{\Lambda(N_{G,J}^{*(0,1)})} + \frac{1}{2} \langle S(e_i^H) e_j^H, K_l \rangle c(e_j^H) c(K_l) \\ &= \nabla_{e_i^H}^{\text{Cliff}_{X_G}} \otimes \text{Id} + \text{Id} \otimes \nabla_{e_i^H}^{\Lambda(N_{G,J}^{*(0,1)})} + \frac{1}{4} \langle \Theta(e_i, e_j), K_l \rangle c(e_j^H) c(K_l). \end{aligned}$$

However, the last term does not preserve  $\Lambda(T^{*(0,1)}X_G)$  and  $\Lambda(N_{G,J}^{*(0,1)})$ .

From (2.60) and (4.119), in general,  $\Phi \mathcal{L}_p \Phi^{-1}$  will not preserve  $\Lambda(T^{*(0,1)}X_G)$  and  $\Lambda(N_{G,J}^{*(0,1)})$  if  $\Theta$  is not null.

Now, we suppose that  $\Theta = 0$  on  $X_G$ .

In this situation, on  $B = U/G \subset X_G \times \mathfrak{g}^*$ , by (2.60), we have

$$(4.120) \quad \Phi \mathcal{L}_p \Phi^{-1} = D_{G,p}^2 - \sum_l (\nabla_{K_l}^{\Lambda(N_{G,J}^{*(0,1)})})^2 + 4\pi^2 |\mathbf{q}|^2 - 2n_0\pi.$$

By Theorem 0.2, Section 3.2 and (3.19), we know that the asymptotic expansion of the Bergman kernel has the following relation for  $(x, Z^\perp) \in N_{G,x}$ ,  $(x', Z'^\perp) \in N_{G,x'}$ ,

$$(4.121) \quad P_p^G((x, Z^\perp), (x', Z'^\perp)) = P_{G,p}(x, x') P_{\mathcal{L}}(Z^\perp, Z'^\perp) + \mathcal{O}(p^{-\infty}).$$

5. COMPUTING THE COEFFICIENT  $\Phi_1$  AND  $P^{(2)}(0, 0)$ 

In this Section,  $(X, \omega, J)$  is a compact Kähler manifold,  $g^{TX}$  is a  $G$ -invariant Riemannian metric on  $TX$  which is compatible with  $J$ .  $(E, h^E)$ ,  $(L, h^L)$  are holomorphic Hermitian vector bundles on  $X$ , and  $\nabla^E, \nabla^L$  are the holomorphic Hermitian connections on  $(E, h^E)$ ,  $(L, h^L)$ . Moreover,

$$\frac{\sqrt{-1}}{2\pi} R^L = \omega.$$

The action of  $G$  is holomorphic and  $G$  acts freely on  $P = \mu^{-1}(0)$ . Thus  $(X_G, \omega_G, J_G)$  is a compact Kähler manifold.

In Sections 5.1-5.5, we suppose that in (0.2),  $\mathbf{J} = J$  on a  $G$ -neighborhood  $U$  of  $P = \mu^{-1}(0)$ .

The main purpose here is to compute the coefficient  $\Phi_1$  in (0.20) and  $P^{(2)}(0, 0)$  in (0.16).

By (0.19) (cf. also Theorem 2.23),

$$(5.1) \quad \Phi_1(x_0) = \int_{Z \in N_{G, x_0}} P_{x_0}^{(2)}(Z, Z) dv_{N_G}(Z).$$

We will first compute explicitly the terms  $\mathcal{O}_1$  and  $\mathcal{O}_2$  involved in  $P^{(2)}$  in (3.32), (3.62), and then compute the integration of  $P^{(2)}$  along the normal spaces to  $X_G$ .

Sometimes the computations seem to be long and tedious, involving many subtle relations between metrics, connections and curvatures near  $X_G$ , but fortunately the final result on  $\Phi_1$  is still of a simple form, as expected.

Throughout the computations below, a key idea is to rewrite all operators by using the creation and annihilation operators  $b_i, b_i^+, b_j^-, b_j^{+}$ , then under the help of (3.9) and Theorem 3.1, we can do the operations and to obtain the crucial Lemmas 5.9, 5.11.

To get the final simple formula (0.25), we still need to prove a highly non-trivial identity (5.130).

The formula for  $P^{(2)}(0, 0)$  in Theorem 0.7 is quite complicate, it involves  $h$ , the volume function of the orbit and the curvature for the principal bundle  $P \rightarrow X_G$ .

This Section is organized as follows. In Section 5.1, we explain various relations of the curvature of the fibration  $P \rightarrow X_G$  and the second fundamental form of  $P$ . In Section 5.2, we obtain the explicit formulas for the operators  $\mathcal{O}_1, \mathcal{O}_2$ . In Section 5.3, we apply the formulas in Section 5.2 and (5.1) to (3.31), and we get a formula for the coefficient  $\Phi_1$ . In Section 5.4, we compute finally  $\Phi_1$ , thus prove Theorem 0.6. In Section 5.5, we compute  $P^{(2)}(0, 0)$  in Theorem 0.7. In Section 5.6, we explain how to reduce the general case to the case  $\mathbf{J} = J$  which has been worked out in Sections 5.1-5.4.

In the whole Section, if there is no other specific notification, when we meet the operation  $|\cdot|^2$ , we will first do this operation, then take the sum of the indices.

**5.1. The second fundamental form of  $P$ .** We use the notations in Sections 2.2, 2.3. Then the normal bundle  $N_G$  of  $X_G$  in  $U/G$  is  $(JTY)_G$ .

Let  $\iota : X_G \rightarrow U/G$  be the natural embedding.

We will apply the notation in Section 1.1 to  $B = U/G$ .

Let  $\nabla^{TX_G}, \nabla^{N_G}$  be connections on  $TX_G, N_G$  on  $X_G$  induced by projections of  $\nabla^{TB}|_{X_G}$ . Then  $\nabla^{TX_G}$  is the Levi-Civita connection on  $(TX_G, g^{TX_G})$ .

Let

$$(5.2) \quad {}^0\nabla^{TB} = \nabla^{TX_G} \oplus \nabla^{N_G}$$

be the connection on  $TB$  on  $X_G$  induced by  $\nabla^{TX_G}, \nabla^{N_G}$  with curvature  ${}^0R^{TB}$ .

Set

$$(5.3) \quad A = \nabla^{TB}|_{X_G} - {}^0\nabla^{TB}.$$

Then  $A$  is a 1-form on  $X_G$  taking values in the skew-adjoint endomorphisms of  $(TB)|_{X_G}$  which exchange  $TX_G$  and  $N_G$ .

We recall the following properties of  $R^{TB}$ : for  $U, V, W, W_2 \in TB$ ,

$$(5.4) \quad \begin{aligned} \langle R^{TB}(U, V)W, W_2 \rangle &= \langle R^{TB}(W, W_2)U, V \rangle, \\ R^{TB}(U, V)W + R^{TB}(V, W)U + R^{TB}(W, U)V &= 0. \end{aligned}$$

On  $X_G$ , let  $\{e_i^0\}$  be an orthonormal frame of  $TX_G$ , let  $\{e_j^\perp\}$  be an orthonormal frame of  $N_G$ , then  $\{e_i\} = \{e_i^0, e_j^\perp\}$  is an orthonormal frame of  $TB$ .

The following result gives detail informations on the torsion  $T$  of the fibration, as well as the second fundamental form  $A$ .

**Theorem 5.1.** *On  $P$ , the restriction of the tensor  $\langle JT(\cdot, J\cdot), \cdot \rangle$  on  $(N_G)^{\otimes 3}$  is symmetric, and*

$$(5.5a) \quad (A(e_i^0)e_j^0)^H = \frac{1}{2}JT(e_i^{0,H}, Je_j^{0,H}),$$

$$(5.5b) \quad T(e_i^{0,H}, e_j^{0,H}) = T((J_G e_i^0)^H, (J_G e_j^0)^H),$$

$$(5.5c) \quad T(e_i^{0,H}, e_j^{\perp,H}) = 2T((J_G e_i^0)^H, Je_j^{\perp,H}),$$

$$(5.5d) \quad \langle T(e_i^{0,H}, e_j^{\perp,H}), Je_k^{\perp,H} \rangle = \langle T(e_i^{0,H}, e_k^{\perp,H}), Je_j^{\perp,H} \rangle,$$

$$(5.5e) \quad \sum_k \langle T(e_k^{\perp,H}, e_j^{\perp,H}), Je_k^{\perp,H} \rangle = 0.$$

*Proof.* Observe first that we have

$$(5.6a) \quad \nabla^{TX} J = 0;$$

$$(5.6b) \quad (J_G e_i^0)^H = Je_i^{0,H} \quad \text{on } P.$$

Let  $Z$  be a smooth section of  $TY$ , then  $JZ \in JTY \subset T^H X$  on  $P$ , by (1.7), (3.1) and (5.6a), on  $P$ , we have

$$(5.7) \quad \begin{aligned} \langle J(A(e_i^0)e_j^0)^H, Z \rangle &= -\langle \nabla_{e_i^{0,H}}^{T^H X} e_j^{0,H}, JZ \rangle = -\langle \nabla_{e_i^{0,H}}^{TX} e_j^{0,H}, JZ \rangle \\ &= \langle \nabla_{e_i^{0,H}}^{TX} (Je_j^{0,H}), Z \rangle = \langle S(e_i^{0,H})Je_j^{0,H}, Z \rangle = -\frac{1}{2} \langle T(e_i^{0,H}, Je_j^{0,H}), Z \rangle. \end{aligned}$$

Thus we get (5.5a), as  $A(e_i^0)e_j^0 \in N_G$ .

Note that  $[Z, e_i^H] \in TY$ , by (1.5), (1.7) and (5.6a),

$$(5.8) \quad \langle T(e_i^H, e_j^H), Z \rangle = 2 \langle \nabla_{e_i^H}^{TX} Z, e_j^H \rangle = 2 \langle \nabla_Z^{TX} e_i^H, e_j^H \rangle = 2 \langle \nabla_Z^{TX} (Je_i^H), Je_j^H \rangle.$$

From (5.6b) and (5.8), we get (5.5b).

From (1.7), (5.8) and  $Je_j^{\perp, H}, Je_k^{\perp, H} \in TY$  on  $P$ , we get

$$(5.9) \quad \langle T(e_i^{0, H}, e_j^{\perp, H}), Z \rangle = 2 \langle S(Z)(Je_i^{0, H}), Je_j^{\perp, H} \rangle = 2 \langle T(Je_i^{0, H}, Je_j^{\perp, H}), Z \rangle.$$

Thus we get (5.5c). By (1.6), (5.9), we get

$$(5.10) \quad \langle T(e_i^{0, H}, e_j^{\perp, H}), Je_k^{\perp, H} \rangle = 2 \langle T(Je_i^{0, H}, Je_j^{\perp, H}), Je_k^{\perp, H} \rangle = \langle T(e_i^{0, H}, e_k^{\perp, H}), Je_j^{\perp, H} \rangle.$$

Thus we get (5.5d). By (1.7), (5.6a) and  $Je_j^{\perp, H} \in TY$  on  $P$ ,

$$(5.11) \quad \begin{aligned} \langle T(e_i^{\perp, H}, Je_j^{\perp, H}), Je_k^{\perp, H} \rangle &= \langle \nabla_{Je_k^{\perp, H}}^{TX} e_i^{\perp, H}, Je_j^{\perp, H} \rangle \\ &= - \langle \nabla_{Je_k^{\perp, H}}^{TX} (Je_i^{\perp, H}), e_j^{\perp, H} \rangle = \langle \nabla_{Je_k^{\perp, H}}^{TX} e_j^{\perp, H}, Je_i^{\perp, H} \rangle = \langle T(e_j^{\perp, H}, Je_i^{\perp, H}), Je_k^{\perp, H} \rangle. \end{aligned}$$

By (1.6) and (5.11),  $\langle JT(\cdot, J\cdot), \cdot \rangle$  is symmetric on the horizontal lift of  $N_G^{\otimes 3}$ .

Note that  $\{Je_k^{\perp, H}\}$  is a  $G$ -invariant orthonormal frame of  $TY$  on  $P$ , by (5.8),

$$(5.12) \quad \langle T(e_i^{\perp, H}, e_j^{\perp, H}), Je_k^{\perp, H} \rangle = 2 \langle \nabla_{Je_k^{\perp, H}}^{TY} (Je_i^{\perp, H}), Je_j^{\perp, H} \rangle.$$

By (1.9) and (5.12), we get (5.5e). The proof of Theorem 5.1 is complete.  $\square$

**Remark 5.2.** From (1.6) and (5.5b),  $\Theta|_{X_G}$  is a  $(1, 1)$ -form on  $X_G$ . Especially, for any complex representation  $V$  of  $G$ ,  $P \times_G V$  is a holomorphic vector bundle on  $X_G$ . Moreover, by (5.5a), for  $U \in TX_G, V \in N_G$ , we have at  $x_0$ ,

$$(5.13) \quad A(U)V = \langle A(U)V, e_j^0 \rangle e_j^0 = - \langle V, A(U)e_j^0 \rangle e_j^0 = \frac{1}{2} \langle T(U, Je_j^0), JV \rangle e_j^0.$$

For  $x_0 \in X_G$ , if  $\{e_j^{\perp}\}$  is a fixed orthonormal basis of  $N_{G, x_0}$  as above, then for  $U \in T_{x_0}X_G$ , we will denote by

$$(5.14) \quad \mathcal{T}_{ijk} = \langle JT(e_i^{\perp}, Je_j^{\perp}), e_k^{\perp} \rangle, \quad \tilde{\mathcal{T}}_{ijk} = \langle JT(e_i^{\perp}, e_j^{\perp}), e_k^{\perp} \rangle, \quad \mathcal{T}_{jk}(U) = \langle JT(U, e_j^{\perp}), e_k^{\perp} \rangle.$$

By Theorem 5.1,  $\mathcal{T}_{ijk}$  is symmetric on  $i, j, k$  and  $\mathcal{T}_{jk} \in T_{x_0}^*X_G$  is symmetric on  $j, k$ ,  $\tilde{\mathcal{T}}_{ijk}$  is anti-symmetric on  $i, j$ .

**Remark 5.3.** From Remark 1.2 and (5.12), we know that  $\langle JT(\cdot, \cdot), \cdot \rangle$  is anti-symmetric on  $(N_G)^{\otimes 3}$  if  $g^{TY}$  is induced by a family of  $Ad$ -invariant metric on  $\mathfrak{g}$ . If  $G$  is abelian, then by (1.12), (5.12),  $T(\cdot, \cdot) = 0$  on  $(N_G)^{\otimes 2}$ , thus  $\tilde{\mathcal{T}}_{ijk} = 0$ .

**5.2. Operators  $\mathcal{O}_1, \mathcal{O}_2$  in (2.103).** We use the notation in Sections 2.6, 3.1, and all tensors will be evaluated at  $x_0 \in X_G$ .

Recall that  $(X, \omega)$  is Kähler and  $\mathbf{J} = J$  on a  $G$ -neighborhood  $U$  of  $P = \mu^{-1}(0)$ , then in (3.5)

$$(5.15) \quad a_i = a_j^\perp = 2\pi.$$

Clearly, on  $U$ , the Levi-Civita connection  $\nabla^{TX}$  preserves  $T^{(1,0)}X$  and  $T^{(0,1)}X$ , and  $\nabla^{T^{(1,0)}X} = P^{T^{(1,0)}X} \nabla^{TX} P^{T^{(1,0)}X}$  is the holomorphic Hermitian connection on  $T^{(1,0)}X$ , while the Clifford connection  $\nabla^{\text{Cliff}}$  on  $\Lambda(T^{*(0,1)}X)$  is  $\nabla^{\Lambda(T^{*(0,1)}X)}$ , the natural connection induced by  $\nabla^{T^{(1,0)}X}$ .

Let  $\bar{\partial}^{L^p \otimes E, *}$  be the canonical formal adjoint of the Dolbeault operator  $\bar{\partial}^{L^p \otimes E}$  on  $\Omega^{0, \bullet}(U, L^p \otimes E)$ . Then the operator  $D_p$  in (2.12) is

$$(5.16) \quad D_p = \sqrt{2} \left( \bar{\partial}^{L^p \otimes E} + \bar{\partial}^{L^p \otimes E, *} \right).$$

Note that  $D_p^2$  preserves the  $\mathbb{Z}$ -grading of  $\Omega^{0, \bullet}(U, L^p \otimes E)$ .

Set

$$(5.17) \quad D_{p,i}^2 = D_p^2|_{\Omega^{0,i}(U, L^p \otimes E)}.$$

Since  $\nabla^{\text{Cliff}}$  preserves the  $\mathbb{Z}$ -grading of  $\Lambda(T^{*(0,1)}X)$ , the operator  $\mathcal{L}_2^t$  in (2.101) also preserves the  $\mathbb{Z}$ -grading on  $\Lambda(T^{*(0,1)}X_0)$ . Moreover,  $\mathcal{L}_2^t$  is invertible on  $\oplus_{q=1}^n \Omega^{0,q}(X_0, L_0^p \otimes E_0)$  for  $t$  small enough.

From Section 3.2, for  $P^{(r)}$  in (0.12),

$$(5.18) \quad P^{(r)} = I_{\mathbb{C} \otimes E_G} P^{(r)} I_{\mathbb{C} \otimes E_G}.$$

Thus we only need to do the computation for  $D_{p,0}^2$ .

In what follows, we compute everything on  $\mathcal{C}^\infty(U, L^p \otimes E)$ .

Take  $x_0 \in X_G$ .

If  $Z \in T_{x_0}B$ ,  $Z = Z^0 + Z^\perp$ ,  $Z^0 \in T_{x_0}X_G$ ,  $Z^\perp \in N_{x_0}$ ,  $|Z^0|, |Z^\perp| \leq \varepsilon$ , in Section 2.6, we identify  $Z$  with  $\exp_{\exp_{x_0}^{X_G}(Z^0)}^B \tau_{Z^0}(Z^\perp)$ . This identification is a diffeomorphism from  $B_{x_0}^{TX_G}(0, \varepsilon) \times B_{x_0}^N(0, \varepsilon)$  into an open neighborhood  $\mathcal{U}(x_0)$  of  $x_0$  in  $B$ , we denote it by  $\Psi$ . Then  $\mathcal{U}(x_0) \cap X_G = B_{x_0}^{TX_G}(0, \varepsilon) \times \{0\}$ .

In what follows, we use indifferently the notation  $B_{x_0}^{TX_G}(0, \varepsilon) \times B_{x_0}^N(0, \varepsilon)$  or  $\mathcal{U}(x_0)$ ,  $x_0$  or  $0, \dots$ .

From now on, we replace  $U/G$  by  $\mathbb{R}^{2n-n_0} \simeq T_{x_0}B$  as in Section 2.6, and we use the notation therein. Especially,

$$(5.19) \quad \nabla_t = t S_t^{-1} \kappa^{1/2} \nabla^{(L^p \otimes E)_B} \kappa^{-1/2} S_t,$$

and  $\mathcal{O}_r$  in (2.103) takes value in  $\text{End}(E_B)$ .

Let  $\{e_i^0\}, \{e_j^\perp\}$  be orthonormal basis of  $T_{x_0}X_G$ ,  $N_{G,x_0}$  respectively. We will also denote  $\Psi_*(e_i^0), \Psi_*(e_j^\perp)$  by  $e_i^0, e_j^\perp$ .

Let  $\{e_i\}$  denote the basis  $\{e_i^0, e_j^\perp\}$ . Thus in our coordinates,

$$(5.20) \quad \frac{\partial}{\partial Z_i^0} = e_i^0, \quad \frac{\partial}{\partial Z_j^\perp} = e_j^\perp.$$

We denote by  $(g^{ij}(Z))$  the inverse of the matrix  $(g_{ij}(Z)) = (g_{ij}^{TB}(Z))$ .

Recall  $\Gamma_{ij}^l$  is the connection form of  $\nabla^{TB}$ , with respect to the frame  $\{e_i\}$ , defined in (2.107). Also recall that  $\mathcal{R}$ ,  $\mathcal{R}^0$  and  $\mathcal{R}^\perp$  are defined in (2.77).

As in (1.14), the moment map  $\mu$  induces a  $G$ -invariant  $\mathcal{C}^\infty$  section  $\tilde{\mu}$  of  $TY$  on  $U$ .

Note also that by (2.48),  $R_\tau^E \in \text{End}(E)$  defines a section of  $\text{End}(E_B)$  on  $B = U/G$ .

Set

$$(5.21) \quad \begin{aligned} \mathcal{L}_3^t(Z) = & -g^{ij}(tZ) \left( \nabla_{t,e_i} \nabla_{t,e_j} - t\Gamma_{ij}^k(tZ) \nabla_{t,e_k} \right) \\ & + t^2 \left( \frac{1}{h} g^{ij} (\nabla_{e_i} \nabla_{e_j} h - \Gamma_{ij}^k \nabla_{e_k} h) \right) (tZ) - t^2 R_\tau^E(tZ) - 2\pi n. \end{aligned}$$

By (2.60), (2.101) and (5.21), we can reformulate (2.102), (2.110), in using the notations in (3.10), as follows,

$$(5.22) \quad \begin{aligned} \nabla_{0,\cdot} &= \nabla_{\cdot} + \frac{1}{2} R_{x_0}^{L_B}(\mathcal{R}, \cdot) = \nabla_{\cdot} - \pi \sqrt{-1} \langle J_{x_0} Z^0, \cdot \rangle_{x_0}, \\ \mathcal{L}_2^0 &= \sum_{j=1}^{n-n_0} b_j b_j^+ + \sum_{j=1}^{n_0} b_j^\perp b_j^{\perp+} = - \sum_j (\nabla_{0,e_j})^2 + 4\pi^2 |Z^\perp|^2 - 2\pi n, \\ \mathcal{L}_2^t(Z) &= \mathcal{L}_3^t(Z) + 4\pi^2 \left| \frac{1}{t} \tilde{\mu} \right|_{g_{TY}}^2(tZ) - \langle 4\pi \sqrt{-1} \tilde{\mu} + t^2 \tilde{\mu}^E, \tilde{\mu}^E \rangle_{g_{TY}}(tZ). \end{aligned}$$

If there is no another specification, we will evaluate our tensors at  $x_0$ , and most of time, we will omit the subscript  $x_0$ .

Set  $h_0 = h_{x_0}$ , and for  $U \in T_{x_0} B$ , set

$$(5.23) \quad \begin{aligned} B(Z, U) &= \frac{1}{2} \sum_{|\alpha|=2} (\partial^\alpha R^{L_B})_{x_0} \frac{Z^\alpha}{\alpha!}(\mathcal{R}, U), \\ I_1 &= -B(Z, e_i) \nabla_{0,e_i} - \frac{1}{2} \nabla_{e_i}(B(Z, e_i)), \\ I_2 &= \left( \left\langle \frac{1}{3} R^{TXG}(\mathcal{R}^0, e_i^0) \mathcal{R}^0 + \nabla_{\mathcal{R}^0}^{TXG}(A(e_i^0) \mathcal{R}^\perp), e_j^0 \right\rangle + \langle e_i^0, \nabla_{\mathcal{R}^0}^{TXG}(A(e_j^0) \mathcal{R}^\perp) \rangle \right. \\ &\quad \left. - 3 \langle A(e_i^0) \mathcal{R}^\perp, A(e_j^0) \mathcal{R}^\perp \rangle + \langle R^{TB}(\mathcal{R}^\perp, e_i^0) \mathcal{R}^\perp, e_j^0 \rangle \right) \nabla_{0,e_i^0} \nabla_{0,e_j^0} \\ &\quad + \left( \langle R^{NG}(\mathcal{R}^0, e_j^0) \mathcal{R}^\perp, e_i^\perp \rangle + \frac{4}{3} \langle R^{TB}(\mathcal{R}^\perp, e_j^0) \mathcal{R}^\perp, e_i^\perp \rangle \right) \nabla_{0,e_i^\perp} \nabla_{0,e_j^0} \\ &\quad + \frac{1}{3} \langle R^{TB}(\mathcal{R}^\perp, e_i^\perp) \mathcal{R}^\perp, e_j^\perp \rangle \nabla_{0,e_i^\perp} \nabla_{0,e_j^\perp}. \end{aligned}$$

Recall that the operator  $\mathcal{L}$  has been defined in (3.10).



Set also

$$\begin{aligned}
 \Gamma_{ii}(\mathcal{R}) &= \frac{2}{3} R_{x_0}^{TXG}(\mathcal{R}^0, e_i^0) e_i^0 + \nabla_{\mathcal{R}^0}^{TB}(A(e_i^0) e_i^0) + R^{TB}(\mathcal{R}^\perp, e_i^0) e_i^0 \\
 &\quad + A(e_i^0) A(e_i^0) \mathcal{R}^\perp + \nabla_{e_i^0}^{TXG}(A(e_i^0) \mathcal{R}^\perp) - A(\mathcal{R}^0) A(e_i^0) e_i^0, \\
 (5.24) \quad K_2(\mathcal{R}) &= \frac{1}{3} \langle R^{TXG}(\mathcal{R}^0, e_i^0) \mathcal{R}^0, e_i^0 \rangle + \langle R^{TB}(\mathcal{R}^\perp, e_i^0) \mathcal{R}^\perp, e_i^0 \rangle \\
 &\quad + \frac{1}{3} \langle R^{TB}(\mathcal{R}^\perp, e_i^\perp) \mathcal{R}^\perp, e_i^\perp \rangle + 2 \left( \sum_i \langle A(e_i^0) e_i^0, \mathcal{R}^\perp \rangle \right)^2 \\
 &\quad - |A(e_i^0) \mathcal{R}^\perp|^2 + 2 \langle \nabla_{\mathcal{R}^0}^{TXG}(A(e_i^0) \mathcal{R}^\perp), e_i^0 \rangle.
 \end{aligned}$$

**Lemma 5.4.** *There exist second order differential operators  $\mathcal{O}'_r$  as in Theorem 2.11 such that for  $|t| \leq 1$ ,*

$$(5.25) \quad \mathcal{L}_3^t = \mathcal{L}_3^0 + \sum_{r=1}^m t^r \mathcal{O}'_r + \mathcal{O}(t^{m+1}),$$

with

$$\begin{aligned}
 (5.26) \quad \mathcal{L}_3^0 &= \mathcal{L} - \sum_{j=1}^{n_0} (\nabla_{e_j^\perp})^2 - 2\pi n_0 = \mathcal{L}_2^0 - 4\pi^2 |Z^\perp|^2, \\
 \mathcal{O}'_1 &= -\frac{2}{3} (\partial_j R^{LB})_{x_0}(\mathcal{R}, e_i) Z_j \nabla_{0, e_i} - \frac{1}{3} (\partial_i R^{LB})_{x_0}(\mathcal{R}, e_i) - 2 \langle A(e_i^0) e_j^0, \mathcal{R}^\perp \rangle \nabla_{0, e_i^0} \nabla_{0, e_j^0}, \\
 \mathcal{O}'_2 &= I_1 + I_2 + \left[ \frac{1}{4} K_2(\mathcal{R}) - \frac{3}{8} \left( \sum_l \langle A(e_l^0) e_l^0, \mathcal{R}^\perp \rangle \right)^2, \mathcal{L}_2^0 \right] \\
 &\quad - 2 \langle A(e_i^0) e_j^0, \mathcal{R}^\perp \rangle \left( \frac{2}{3} (\partial_k R^{LB})_{x_0}(\mathcal{R}, e_j^0) Z_k \nabla_{0, e_i^0} + \frac{1}{3} (\partial_j^0 R^{LB})_{x_0}(\mathcal{R}, e_i^0) \right) \\
 &\quad + \langle \Gamma_{ii}(\mathcal{R}), e_j \rangle \nabla_{0, e_j} - \frac{1}{2} \langle A(e_l^0) e_l^0, \mathcal{R}^\perp \rangle \nabla_{A(e_k^0) e_k^0} + 2 \langle A(e_i^0) e_j^0, \mathcal{R}^\perp \rangle \nabla_{A(e_i^0) e_j^0} \\
 &\quad + \frac{2}{3} \langle R^{TB}(\mathcal{R}^\perp, e_i^\perp) e_i^\perp, e_j \rangle \nabla_{0, e_j} - R_{x_0}^{EB}(\mathcal{R}, e_i) \nabla_{0, e_i} - R_{\tau, x_0}^{EB} \\
 &\quad - \frac{1}{9} \sum_i \left[ \sum_j (\partial_j R^{LB})_{x_0}(\mathcal{R}, e_i) Z_j \right]^2 + \frac{1}{h_0} (\nabla_{e_j} \nabla_{e_j} h - \nabla_{A(e_i^0) e_i^0} h)_{x_0}.
 \end{aligned}$$

*Proof.* By (2.104) and (5.19),

$$\begin{aligned}
 (5.27) \quad \nabla_{t, e_i} &= \kappa^{1/2} (tZ) \left( \nabla_{e_i} + \left( \frac{1}{2} R_{x_0}^{LB} + \frac{t}{3} (\partial_k R^{LB})_{x_0} Z_k \right. \right. \\
 &\quad \left. \left. + \frac{t^2}{4} \sum_{|\alpha|=2} (\partial^\alpha R^{LB})_{x_0} \frac{Z^\alpha}{\alpha!} + \frac{t^2}{2} R_{x_0}^{EB} \right) (\mathcal{R}, e_i) + \mathcal{O}(t^3) \right) \kappa^{-1/2} (tZ).
 \end{aligned}$$

To get (5.26), we could use (2.93)-(2.97), while here we will get it directly from the local computation.

By [1, Prop. 1.28] (cf. [26, (1.31)]) and (2.104),

$$(5.28) \quad \begin{aligned} \langle e_i^0, e_j^0 \rangle_{Z^0} &= \delta_{ij} + \frac{1}{3} \langle R_{x_0}^{TXG}(\mathcal{R}^0, e_i^0) \mathcal{R}^0, e_j^0 \rangle_{x_0} + \mathcal{O}(|Z^0|^3), \\ (\nabla_{e_k^0}^{NG} \nabla_{e_i^0}^{NG} e_j^\perp)_{x_0} &= \frac{1}{2} R_{x_0}^{NG}(e_k^0, e_i^0) e_j^\perp. \end{aligned}$$

Moreover, for  $W, V \in N_{x_0}$ ,  $\gamma_s(t) = (Z^0, t(W + sV))$  is a family of geodesics from  $(Z^0, 0)$ .

Set  $Y = \frac{\partial}{\partial t} \gamma_s(t)$ ,  $X(\gamma_s(t)) = \frac{\partial}{\partial s} \gamma_s(t) = tV$ .

Since  $\nabla_Y^{TB} Y = 0$ ,  $\nabla_Y^{TB} X - \nabla_X^{TB} Y = [Y, X] = \gamma_*[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}] = 0$ , we get

$$(5.29) \quad 0 = \nabla_X^{TB} \nabla_Y^{TB} Y = \nabla_Y^{TB} \nabla_Y^{TB} X - R^{TB}(Y, X)Y.$$

Take  $V = e_i^\perp$ , we get at  $s = t = 0$ ,

$$(5.30) \quad (\nabla_W^{TB} \nabla_W^{TB} e_i^\perp)_{Z^0} = \frac{1}{3} \nabla_Y^{TB} \nabla_Y^{TB} \nabla_Y^{TB} X = \frac{1}{3} R^{TB}(W, e_i^\perp)W.$$

Under our coordinate, we have

$$(5.31) \quad \begin{aligned} (\nabla_{e_j^\perp}^{TB} e_i^\perp)_{x_0} &= (\nabla_{e_i^0}^{TXG} e_j^0)_{x_0} = (\nabla_{e_j^0}^{NG} e_i^\perp)_{x_0} = 0, \quad (\nabla_{e_i^0}^{TB} e_j^0)_{x_0} = A_{x_0}(e_i^0) e_j^0, \\ (\nabla_{e_j^\perp}^{TB} e_i^0)_{x_0} &= (\nabla_{e_i^0}^{TB} e_j^\perp)_{x_0} = A_{x_0}(e_i^0) e_j^\perp, \\ (\nabla_{\mathcal{R}^\perp}^{TB} e_i^\perp)_Z &= 0, \\ (\nabla_{e_j^\perp}^{TB} e_i^\perp)_{Z^0} &= (\nabla_{\mathcal{R}^0}^{NG} e_i^\perp)_{Z^0} = 0. \end{aligned}$$

Moreover, by (5.4), (5.28), (5.30) and (5.31) (comparing with [26, (1.31)]), we have at  $x_0$  that

$$(5.32) \quad \begin{aligned} \nabla_{e_k^\perp}^{TB} \nabla_{e_j^\perp}^{TB} e_i^\perp &= \frac{1}{3} R^{TB}(e_k^\perp, e_j^\perp) e_i^\perp + \frac{1}{3} R^{TB}(e_k^\perp, e_i^\perp) e_j^\perp, \\ \nabla_{e_k^0}^{TB} \nabla_{e_j^\perp}^{TB} e_i^\perp &= 0, \\ \nabla_{e_k^\perp}^{TB} \nabla_{e_j^\perp}^{TB} e_i^0 &= \nabla_{e_k^\perp}^{TB} \nabla_{e_i^0}^{TB} e_j^\perp = R^{TB}(e_k^\perp, e_i^0) e_j^\perp, \\ \nabla_{e_k^0}^{TB} \nabla_{e_j^\perp}^{TB} e_i^0 &= \nabla_{e_k^0}^{TB} \nabla_{e_i^0}^{TB} e_j^\perp = \nabla_{e_k^0}^{NG} \nabla_{e_i^0}^{NG} e_j^\perp + A(e_k^0) A(e_i^0) e_j^\perp + \nabla_{e_k^0}^{TXG}(A(e_i^0) e_j^\perp) \\ &= \frac{1}{2} R^{NG}(e_k^0, e_i^0) e_j^\perp + A(e_k^0) A(e_i^0) e_j^\perp + \nabla_{e_k^0}^{TXG}(A(e_i^0) e_j^\perp), \\ \nabla_{e_j^\perp}^{TB} \nabla_{e_k^0}^{TB} e_i^0 &= R^{TB}(e_j^\perp, e_k^0) e_i^0 + \nabla_{e_k^0}^{TB} \nabla_{e_j^\perp}^{TB} e_i^0, \\ \nabla_{e_k^0}^{TB} \nabla_{e_j^0}^{TB} e_i^0 &= \nabla_{e_k^0}^{TXG} \nabla_{e_j^0}^{TXG} e_i^0 + \nabla_{e_k^0}^{TB}(A(e_j^0) e_i^0) \\ &= \frac{1}{3} R^{TXG}(e_k^0, e_j^0) e_i^0 + \frac{1}{3} R^{TXG}(e_k^0, e_i^0) e_j^0 + \nabla_{e_k^0}^{TB}(A(e_j^0) e_i^0). \end{aligned}$$

In the following, for a tensor  $\psi$  and the covariant derivative  $\nabla^B$  acting on  $\psi$  induced by  $\nabla^{TB}$ , we denote by

$$(\nabla^B \nabla^B \psi)_{(c_j e_j, c'_k e_k)} = c_j c'_k (\nabla_{e_j}^B \nabla_{e_k}^B \psi)_{x_0}.$$

From (5.32), we get at  $x_0$  the following formula which will be used in (5.37), (5.38), (5.55), (5.56) and (5.157),

$$\begin{aligned}
(\nabla^{TB} \nabla^{TB} e_i^0)_{(\mathcal{R}^0, \mathcal{R}^0)} &= \frac{1}{3} R^{TXG}(\mathcal{R}^0, e_i^0) \mathcal{R}^0 + \nabla_{\mathcal{R}^0}^{TB} (A(e_j^0) e_i^0) Z_j^0, \\
(\nabla^{TB} \nabla^{TB} e_i^0)_{(\mathcal{R}^0, \mathcal{R}^\perp)} &= \frac{1}{2} R^{NG}(\mathcal{R}^0, e_i^0) \mathcal{R}^\perp + A(\mathcal{R}^0) A(e_i^0) \mathcal{R}^\perp + \nabla_{\mathcal{R}^0}^{TXG} (A(e_i^0) \mathcal{R}^\perp), \\
(\nabla^{TB} \nabla^{TB} e_i^0)_{(\mathcal{R}^\perp, \mathcal{R}^\perp)} &= R^{TB}(\mathcal{R}^\perp, e_i^0) \mathcal{R}^\perp, \\
(5.33) \quad (\nabla^{TB} \nabla^{TB} e_j^\perp)_{(\mathcal{R}^0, \mathcal{R}^0)} &= A(\mathcal{R}^0) A(\mathcal{R}^0) e_j^\perp + \nabla_{\mathcal{R}^0}^{TXG} (A(e_k^0) e_j^\perp) Z_k^0, \\
(\nabla^{TB} \nabla^{TB} e_j^\perp)_{(\mathcal{R}^0, \mathcal{R}^\perp)} &= 0, \\
(\nabla^{TB} \nabla^{TB} e_j^\perp)_{(\mathcal{R}^\perp, \mathcal{R}^\perp)} &= \frac{1}{3} R^{TB}(\mathcal{R}^\perp, e_j^\perp) \mathcal{R}^\perp, \\
(\nabla^{TB} \nabla^{TB} e_j)_{(\mathcal{R}^\perp, \mathcal{R}^0)} &= (\nabla^{TB} \nabla^{TB} e_j)_{(\mathcal{R}^0, \mathcal{R}^\perp)} + R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0) e_j.
\end{aligned}$$

Note that by (5.31),  $\nabla_{\mathcal{R}}^{TB} (A_{x_0}(e_i^0) e_i^0) = A(\mathcal{R}^0) A_{x_0}(e_i^0) e_i^0$ .

From (5.31), (5.32), we get

$$\begin{aligned}
(\nabla_{e_i^\perp}^{TB} e_i^\perp)_Z &= \frac{2}{3} R^{TB}(\mathcal{R}^\perp, e_i^\perp) e_i^\perp + \mathcal{O}(|Z|^2), \\
(5.34) \quad (\nabla_{e_i^0}^{TB} e_i^0)_Z &= A_{x_0}(e_i^0) e_i^0 - \nabla_{\mathcal{R}}^{TB} (A_{x_0}(e_i^0) e_i^0) + \frac{2}{3} R^{TXG}(\mathcal{R}^0, e_i^0) e_i^0 + \nabla_{\mathcal{R}^0}^{TB} (A(e_i^0) e_i^0) \\
&\quad + A(e_i^0) A(e_i^0) \mathcal{R}^\perp + \nabla_{e_i^0}^{TXG} (A(e_i^0) \mathcal{R}^\perp) + R^{TB}(\mathcal{R}^\perp, e_i^0) e_i^0 + \mathcal{O}(|Z|^2) \\
&= A_{x_0}(e_i^0) e_i^0 + \Gamma_{ii}(\mathcal{R}) + \mathcal{O}(|Z|^2),
\end{aligned}$$

Thus by (5.31), (5.32) and (5.33), at  $x_0$ ,

$$\begin{aligned}
(5.35) \quad \nabla_{\mathcal{R}^0} \nabla_{\mathcal{R}^\perp} \langle e_j^\perp, e_i^0 \rangle &= \langle \nabla_{\mathcal{R}^0}^{TB} e_j^\perp, \nabla_{\mathcal{R}^\perp}^{TB} e_i^0 \rangle + \langle e_j^\perp, \nabla_{\mathcal{R}^0}^{TB} \nabla_{\mathcal{R}^\perp}^{TB} e_i^0 \rangle \\
&= \frac{1}{2} \langle R^{NG}(\mathcal{R}^0, e_i^0) \mathcal{R}^\perp, e_j^\perp \rangle.
\end{aligned}$$

On the other hand, we have the following expansion for  $\langle e_j, e_i \rangle_Z$ ,

$$\begin{aligned}
(5.36) \quad \langle e_i, e_j \rangle_Z &= \langle e_i, e_j \rangle_{Z^0} + (\nabla_{\mathcal{R}^\perp} \langle e_i, e_j \rangle)_{Z^0} + \frac{1}{2} (\nabla \nabla \langle e_i, e_j \rangle)_{(\mathcal{R}^\perp, \mathcal{R}^\perp), x_0} + \mathcal{O}(|Z|^3) \\
&= \langle e_i, e_j \rangle_{Z^0} + (\nabla_{\mathcal{R}^\perp} \langle e_i, e_j \rangle)_{x_0} + (\nabla_{\mathcal{R}^0} \nabla_{\mathcal{R}^\perp} \langle e_i, e_j \rangle)_{x_0} + \langle \nabla_{\mathcal{R}^\perp}^{TB} e_i, \nabla_{\mathcal{R}^\perp}^{TB} e_j \rangle_{x_0} \\
&\quad + \frac{1}{2} \langle (\nabla^{TB} \nabla^{TB} e_i)_{(\mathcal{R}^\perp, \mathcal{R}^\perp)}, e_j \rangle + \frac{1}{2} \langle e_i, (\nabla^{TB} \nabla^{TB} e_j)_{(\mathcal{R}^\perp, \mathcal{R}^\perp)} \rangle + \mathcal{O}(|Z|^3).
\end{aligned}$$

Thus by (5.28), (5.31), (5.33) and (5.35)-(5.36),

$$\begin{aligned}
(5.37) \quad \langle e_i^0, e_j^0 \rangle_Z &= \delta_{ij} - 2 \langle A_{x_0}(e_i^0) e_j^0, \mathcal{R}^\perp \rangle + \frac{1}{3} \langle R^{TXG}(\mathcal{R}^0, e_i^0) \mathcal{R}^0, e_j^0 \rangle \\
&\quad + \langle \nabla_{\mathcal{R}^0}^{TXG} (A(e_i^0) \mathcal{R}^\perp), e_j^0 \rangle + \langle e_i^0, \nabla_{\mathcal{R}^0}^{TXG} (A(e_j^0) \mathcal{R}^\perp) \rangle \\
&\quad + \langle A(e_i^0) \mathcal{R}^\perp, A(e_j^0) \mathcal{R}^\perp \rangle + \langle R^{TB}(\mathcal{R}^\perp, e_i^0) \mathcal{R}^\perp, e_j^0 \rangle + \mathcal{O}(|Z|^3),
\end{aligned}$$

and

$$(5.38) \quad \begin{aligned} \langle e_i^0, e_j^\perp \rangle_Z &= \frac{1}{2} \langle R^{NG}(\mathcal{R}^0, e_i^0) \mathcal{R}^\perp, e_j^\perp \rangle + \frac{2}{3} \langle R^{TB}(\mathcal{R}^\perp, e_i^0) \mathcal{R}^\perp, e_j^\perp \rangle + \mathcal{O}(|Z|^3), \\ \langle e_i^\perp, e_j^\perp \rangle_Z &= \delta_{ij} + \frac{1}{3} \langle R^{TB}(\mathcal{R}^\perp, e_i^\perp) \mathcal{R}^\perp, e_j^\perp \rangle + \mathcal{O}(|Z|^3). \end{aligned}$$

From (5.24), (5.37) and (5.38), we get

$$(5.39) \quad \begin{aligned} \det g_{ij}(Z) &= 1 - 2 \langle A_{x_0}(e_i^0) e_i^0, \mathcal{R}^\perp \rangle + K_2(\mathcal{R}) + \mathcal{O}(|Z|^3), \\ \kappa^{\frac{1}{2}}(tZ) &= (\det g_{ij})^{1/4}(tZ) \\ &= 1 - \frac{t}{2} \langle A(e_i^0) e_i^0, \mathcal{R}^\perp \rangle - \frac{3t^2}{8} \left( \sum_i \langle A(e_i^0) e_i^0, \mathcal{R}^\perp \rangle \right)^2 + \frac{t^2}{4} K_2(\mathcal{R}) + \mathcal{O}(t^3), \\ \kappa^{-\frac{1}{2}}(tZ) &= 1 + \frac{t}{2} \langle A(e_i^0) e_i^0, \mathcal{R}^\perp \rangle + \frac{5t^2}{8} \left( \sum_i \langle A(e_i^0) e_i^0, \mathcal{R}^\perp \rangle \right)^2 - \frac{t^2}{4} K_2(\mathcal{R}) + \mathcal{O}(t^3). \end{aligned}$$

Moreover, as a  $2(n - n_0) \times 2(n - n_0)$ -matrix, we have

$$(5.40) \quad \begin{aligned} \left( (\delta_{ij} - 2 \langle A_{x_0}(e_i^0) e_j^0, \mathcal{R}^\perp \rangle) \right)^{-1} &= (\delta_{ij} + 2 \langle A_{x_0}(e_i^0) e_j^0, \mathcal{R}^\perp \rangle) \\ &\quad + 4 \left( \langle A_{x_0}(e_i^0) \mathcal{R}^\perp, A_{x_0}(e_j^0) \mathcal{R}^\perp \rangle \right) + \mathcal{O}(|Z|^3). \end{aligned}$$

Note that from (3.9), (5.22),

$$(5.41) \quad [\langle A(e_i^0) e_i^0, \mathcal{R}^\perp \rangle, \mathcal{L}_2^0] = 2 \langle A(e_i^0) e_i^0, e_k^\perp \rangle \nabla_{0, e_k^\perp}.$$

Thus from (5.24), (5.27), (5.34), (5.37)-(5.39), the coefficients of  $t$ ,  $t^2$  in the expansion of  $g^{ij}(tZ) t \Gamma_{ij}^k(tZ) \nabla_{t, e_k} = t g^{ij}(tZ) \nabla_{t, (\nabla_{e_i}^{TB} e_j)(tZ)}$  are

$$(5.42) \quad \begin{aligned} &\langle A(e_i^0) e_i^0, e_k^\perp \rangle \nabla_{0, e_k^\perp}; \\ &2 \langle A(e_i^0) e_j^0, \mathcal{R}^\perp \rangle \nabla_{A(e_i^0) e_j^0} + \langle \Gamma_{ii}(\mathcal{R}), e_j \rangle \nabla_{0, e_j} + \frac{2}{3} \langle R^{TB}(\mathcal{R}^\perp, e_i^\perp) e_i^\perp, e_j \rangle \nabla_{0, e_j} \\ &\quad - \left[ \frac{1}{2} \langle A(e_l^0) e_l^0, \mathcal{R}^\perp \rangle, \nabla_{A(e_i^0) e_i^0} \right] + \frac{1}{3} (\partial_k R^{LB})_{x_0} Z_k(\mathcal{R}, A(e_i^0) e_i^0). \end{aligned}$$

By (5.21), (5.27) and (5.37)-(5.42), the coefficient of  $t$  in the expansion of  $\mathcal{L}_3^t$  is  $\mathcal{O}'_1$  in (5.26).

We denote by  $[A, B]_+ = AB + BA$ .

By (5.21), (5.27), (5.34) and (5.37)-(5.40), the coefficient of  $t^2$  in the expansion of  $\mathcal{L}_3^t - (g^{ij}t\Gamma_{ij}^k)(tZ)\nabla_{t,e_k}$  is

$$\begin{aligned}
(5.43) \quad I_2 - 2 \langle A(e_i^0)e_j^0, \mathcal{R}^\perp \rangle & \left[ \frac{1}{3} \nabla_{0,e_i^0} (\partial_k R^{L_B})_{x_0}(\mathcal{R}, e_j^0) Z_k \right. \\
& + \frac{1}{3} (\partial_k R^{L_B})_{x_0}(\mathcal{R}, e_i^0) Z_k \nabla_{0,e_j^0} - \frac{1}{2} [\langle A(e_l^0)e_l^0, \mathcal{R}^\perp \rangle, \nabla_{0,e_i^0} \nabla_{0,e_j^0}] \\
& + I_1 + \left[ \frac{1}{2} \langle A(e_l^0)e_l^0, \mathcal{R}^\perp \rangle, \left[ \frac{1}{3} (\partial_k R^{L_B})_{x_0}(\mathcal{R}, e_i) Z_k, \nabla_{0,e_i} \right]_+ \right] \\
& + \left[ \frac{1}{4} K_2(\mathcal{R}) - \frac{3}{8} \left( \sum_l \langle A(e_l^0)e_l^0, \mathcal{R}^\perp \rangle \right)^2, \mathcal{L}_2^0 \right] \\
& - \frac{1}{4} [\langle A(e_l^0)e_l^0, \mathcal{R}^\perp \rangle, \mathcal{L}_2^0] \langle A(e_k^0)e_k^0, \mathcal{R}^\perp \rangle - R_{x_0}^{E_B}(\mathcal{R}, e_i) \nabla_{0,e_i} \\
& - \frac{1}{9} \sum_i \left[ \sum_j (\partial_j R^{L_B})_{x_0}(\mathcal{R}, e_i) Z_j \right]^2 - R_{\tau,x_0}^E + \frac{1}{h_0} (\nabla_{e_j} \nabla_{e_j} h - \nabla_{A(e_i^0)e_i^0} h)_{x_0}.
\end{aligned}$$

Here  $I_2$  is from the coefficient of  $t^2$  in the expansion of  $g^{ij}$ , the second term is the product of the coefficients of  $t^1$  in the expansion of  $g^{ij}$  and  $\nabla_{t,e_i} \nabla_{t,e_j}$ ;  $I_1$  is from the coefficient of  $t^2$  in the expansion of  $R^{L_B}$ , the fourth term is from the product of the coefficients of  $t^1$  in  $\kappa^{1/2}, \kappa^{-1/2}$  and in  $\kappa^{-1/2} \nabla_{t,e_i} \nabla_{t,e_i} \kappa^{1/2}$  (cf. (5.27)), the fifth and sixth terms is from the coefficients of  $t^2$  in the expansion of  $\kappa^{1/2}, \kappa^{-1/2}$  and the product of the coefficients of  $t^1$  in the expansion of  $\kappa^{1/2}$  and  $\kappa^{-1/2}$ ; the seventh term is from  $R^{E_B}$ , and the eighth term is from the product of the coefficients of  $t^1$  in the expansion of  $R^{L_B}$ .

Certainly,

$$(5.44) \quad \frac{1}{6} \left[ \langle A(e_l^0)e_l^0, \mathcal{R}^\perp \rangle, [(\partial_k R^{L_B})_{x_0}(\mathcal{R}, e_i) Z_k, \nabla_{0,e_i}]_+ \right] = -\frac{1}{3} (\partial_k R^{L_B})_{x_0}(\mathcal{R}, A(e_l^0)e_l^0) Z_k.$$

By (5.41), (5.42), (5.44) and by the fact that  $A(e_i^0)e_j^0$  is symmetric on  $i, j$ , we see that the coefficient of  $t^2$  in the expansion of  $\mathcal{L}_3^t$  is  $\mathcal{O}_2^t$  in (5.26).  $\square$

To simplify the notation, we will often denote by  $e_i$  the lift  $e_i^H$  of  $e_i$ .

**Lemma 5.5.** *The following identities hold,*

$$(5.45a) \quad (\partial_i R^{LB})_{x_0}(\mathcal{R}, e_l) Z_i = -3\sqrt{-1}\pi \langle JT(\mathcal{R}, e_l) - JT(\mathcal{R}^0, P^{TX_G} e_l), \mathcal{R}^\perp \rangle,$$

$$(5.45b) \quad \begin{aligned} \frac{\sqrt{-1}}{\pi} B(Z, e_l^0) &= \frac{1}{6} \langle R^{TX_G}(\mathcal{R}^0, J\mathcal{R}^0) \mathcal{R}^0, e_l^0 \rangle - \frac{5}{4} \langle J\mathcal{R}^\perp, \nabla_{\mathcal{R}}^{TY}(T(e_i, e_l^0)) Z_i \rangle \\ &+ \frac{1}{2} \langle 2\nabla_{\mathcal{R}^0}^{TX_G}(A(e_l^0) e_j^\perp) Z_j^\perp + R^{TB}(\mathcal{R}^\perp, e_l^0) \mathcal{R}^\perp + R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0) e_l^0, J\mathcal{R}^0 \rangle \\ &- \frac{1}{2} \langle 3\nabla_{\mathcal{R}^0}^{TX_G}(A(e_i^0) e_j^\perp) Z_i^0 Z_j^\perp + 2R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0) \mathcal{R}^\perp + R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0) \mathcal{R}^0, J e_l^0 \rangle \\ &+ \frac{1}{2} \left\langle J\mathcal{R}^\perp, T(\mathcal{R}^0 - \frac{1}{4} \mathcal{R}, e_i^0) \right\rangle \langle J\mathcal{R}^\perp, T(e_i^0, J e_l^0) \rangle \\ &+ \frac{1}{8} \langle T(\mathcal{R}^0, \mathcal{R}^\perp), T(e_l^0, J\mathcal{R}^0) \rangle + \frac{1}{8} \langle T(\mathcal{R}^0, J\mathcal{R}^0), T(\mathcal{R}^\perp, e_l^0) \rangle \\ &- \frac{1}{8} \langle T(\mathcal{R}^\perp, J\mathcal{R}^0), T(\mathcal{R}, e_l^0) \rangle + \frac{1}{2} \langle T(\mathcal{R}^\perp, J\mathcal{R}^\perp), T(\mathcal{R}, e_l^0) \rangle \\ &- \frac{1}{8} \langle T(e_l^0, J\mathcal{R}^0), e_j^\perp \rangle \langle J\mathcal{R}^\perp, T(\mathcal{R}^\perp, e_j^\perp) \rangle. \end{aligned}$$

*Proof.* By (1.6), (1.14), (1.18) and (2.14),

$$(5.46) \quad \begin{aligned} \frac{\sqrt{-1}}{2\pi} R^{LB}(e_k, e_l) &= \langle J e_k^H, e_l^H \rangle + \mu(\Theta)(e_k, e_l) \\ &= \langle J e_k^H, e_l^H \rangle + \langle \tilde{\mu}, T(e_k, e_l) \rangle. \end{aligned}$$

Thus by (3.33), (5.5a) and (5.6a), we get at  $x_0$  the following formulas which will be used in (5.61),

$$(5.47) \quad \tilde{\mu}_{x_0} = 0, \quad (\nabla_{\mathcal{R}}^{TY} \tilde{\mu})_{x_0} = -J\mathcal{R}^\perp, \quad (\nabla_{\mathcal{R}}^{TY} \nabla_{\mathcal{R}}^{TY} \tilde{\mu})_{(\mathcal{R}, \mathcal{R})} = T(\mathcal{R}^\perp, J\mathcal{R}^\perp).$$

By (3.36) and  $\mu = 0$  on  $P$ , we have at  $x_0$ ,

$$(5.48) \quad \begin{aligned} (\nabla_{e_i} \langle \tilde{\mu}, T(e_k, e_l) \rangle)_{x_0} &= \left\langle \nabla_{e_i^H}^{TY} \tilde{\mu}, T(e_k, e_l) \right\rangle + \left\langle \tilde{\mu}, \nabla_{e_i^H}^{TY} (T(e_k, e_l)) \right\rangle \\ &= \langle JT(e_k, e_l), e_i \rangle. \end{aligned}$$

By (3.40), (5.6a) and (5.31), we have

$$(5.49) \quad \begin{aligned} (\nabla_{e_i^H} \langle J e_k^H, e_l^H \rangle)_{x_0} &= \left\langle J \nabla_{e_i^H}^{TX} e_k^H, e_l^H \right\rangle_{x_0} + \left\langle J e_k^H, \nabla_{e_i^H}^{TX} e_l^H \right\rangle_{x_0} \\ &= -\frac{1}{2} \langle JT(e_i, e_k), e_l \rangle - \frac{1}{2} \langle J e_k, T(e_i, e_l) \rangle \\ &+ \langle JA(P^{TX_G} e_i) P^{NG} e_k + JA(P^{TX_G} e_k) P^{NG} e_i, P^{TX_G} e_l \rangle \\ &+ \langle J P^{TX_G} e_k, A(P^{TX_G} e_i) P^{NG} e_l + A(P^{TX_G} e_l) P^{NG} e_i \rangle. \end{aligned}$$

By (5.5a), (5.46), (5.48) and (5.49), for  $U \in T_{x_0}B$ ,

$$(5.50) \quad \frac{\sqrt{-1}}{2\pi}(\partial_U R^{L_B})_{x_0}(U, e_l) = \frac{3}{2} \langle JT(U, e_l), U \rangle - 2 \langle A(P^{TX_G}U)P^{N_G}U, JP^{TX_G}e_l \rangle \\ + \langle JP^{TX_G}U, A(P^{TX_G}U)P^{N_G}e_l + A(P^{TX_G}e_l)P^{N_G}U \rangle \\ = \frac{3}{2} \langle JT(U, e_l) - JT(P^{TX_G}U, P^{TX_G}e_l), U \rangle.$$

Note that  $(JTY)_G = N_G$  on  $X_G$ , by (5.50), we get (5.45a).

By (5.23) and (5.46), one gets at  $x_0$ ,

$$(5.51) \quad \frac{\sqrt{-1}}{\pi}B(Z, e_l) = \frac{1}{2} \left( \nabla \nabla \langle J e_k, e_l \rangle + \nabla \nabla \langle \tilde{\mu}, T(e_k, e_l) \rangle \right)_{(\mathcal{R}, \mathcal{R})} Z_k.$$

From (5.6a) we have

$$(5.52) \quad \left( \nabla \nabla \langle J e_k^H, e_l^H \rangle \right)_{(\mathcal{R}, \mathcal{R})} Z_k = \langle J\mathcal{R}, (\nabla^{TX} \nabla^{TX} e_l^H)_{(\mathcal{R}, \mathcal{R})} \rangle \\ + \langle J(\nabla^{TX} \nabla^{TX} e_k^H)_{(\mathcal{R}, \mathcal{R})}, e_l^H \rangle Z_k + 2 \langle J \nabla_{\mathcal{R}}^{TX} e_k^H, \nabla_{\mathcal{R}}^{TX} e_l^H \rangle Z_k.$$

From (1.2), (5.31), one finds at  $x_0$  that

$$(5.53) \quad J\mathcal{R}^\perp \in TY, \quad J\mathcal{R}^0 \in TX_G, \\ \nabla_{\mathcal{R}}^{TB} e_i^0 = A(e_i^0)\mathcal{R}, \quad \nabla_{\mathcal{R}}^{TB} e_i^\perp = A(\mathcal{R}^0)e_i^\perp, \\ (\nabla_{e_j^H}^{TX} e_i^H)Z_i Z_j = (\nabla_{e_j}^{TB} e_i)^H Z_i Z_j = 2A(\mathcal{R}^0)\mathcal{R}^\perp + A(\mathcal{R}^0)\mathcal{R}^0.$$

Now by (3.40),

$$(5.54) \quad (\nabla_{e_j^H}^{TX} \nabla_{e_i^H}^{TX} e_k^H)_{x_0} = \nabla_{e_j^H}^{TX} \nabla_{e_i^H}^{TX} e_k^H - \frac{1}{2}T(e_j^H, \nabla_{e_i^H}^{TX} e_k^H) - \frac{1}{2}\nabla_{e_j^H}^{TX}(T(e_i^H, e_k^H)).$$

By (5.33), we get

$$(5.55) \quad (\nabla^{TB} \nabla^{TB} e_k)_{(\mathcal{R}, \mathcal{R})} Z_k = \nabla_{\mathcal{R}^0}^{TB} (A(e_j^0)e_i^0)Z_j^0 Z_i^0 + 3A(\mathcal{R}^0)A(\mathcal{R}^0)\mathcal{R}^\perp \\ + 3\nabla_{\mathcal{R}^0}^{TX_G} (A(e_i^0)\mathcal{R}^\perp)Z_i^0 + 2R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)\mathcal{R}^\perp + R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)\mathcal{R}^0.$$

From (5.33), (5.53), (5.54), (5.55), the anti-symmetric property of the torsion tensor  $T$  and the fact that  $A$  exchanges  $TX_G$  and  $N_G$ , we get

$$(5.56) \quad \langle J\mathcal{R}, (\nabla^{TX} \nabla^{TX} e_l^{0,H})_{(\mathcal{R}, \mathcal{R})} \rangle = \left\langle \frac{1}{3}R^{TX_G}(\mathcal{R}^0, e_l^0)\mathcal{R}^0 + \nabla_{\mathcal{R}^0}^{TB}(A(e_j^0)e_l^0)Z_j^0, J\mathcal{R}^0 \right\rangle \\ + \langle 2\nabla_{\mathcal{R}^0}^{TX_G}(A(e_l^0)e_j^\perp)Z_j^\perp + R^{TB}(\mathcal{R}^\perp, e_l^0)\mathcal{R}^\perp + R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)e_l^0, J\mathcal{R}^0 \rangle \\ - \frac{1}{2} \langle J\mathcal{R}^\perp, T(\mathcal{R}, A(e_l^0)\mathcal{R}) \rangle - \frac{1}{2} \langle J\mathcal{R}, \nabla_{\mathcal{R}}^{TX}(T(e_i, e_l^0))Z_i \rangle, \\ \langle J(\nabla^{TX} \nabla^{TX} e_k^H)_{(\mathcal{R}, \mathcal{R})}, e_l^{0,H} \rangle Z_k = \langle 2JR^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)\mathcal{R}^\perp + JR^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)\mathcal{R}^0, e_l^0 \rangle \\ + \langle J\nabla_{\mathcal{R}^0}^{TB}(A(e_j^0)e_i^0)Z_j^0 Z_i^0 + 3J\nabla_{\mathcal{R}^0}^{TX_G}(A(e_i^0)e_j^\perp)Z_i^0 Z_j^\perp, e_l^0 \rangle.$$

Note that from (1.8), (5.3), (5.5a), (5.53) and  $A$  exchanges  $TX_G$  and  $N_G$ ,

$$\begin{aligned}
 (5.57) \quad \langle J\mathcal{R}, \nabla_{\mathcal{R}}^{TX}(T(e_i, e_l^0))Z_i \rangle &= \langle J\mathcal{R}^\perp, \nabla_{\mathcal{R}}^{TY}(T(e_i, e_l^0))Z_i \rangle \\
 &\quad + \frac{1}{2} \langle T(\mathcal{R}, J\mathcal{R}^0), T(\mathcal{R}, e_l^0) \rangle, \\
 \langle J\nabla_{\mathcal{R}^0}^{TB}(A(e_j^0)e_i^0)Z_j^0Z_i^0, e_l^0 \rangle &= -\langle A(\mathcal{R}^0)A(\mathcal{R}^0)\mathcal{R}^0, Je_l^0 \rangle \\
 &= -\frac{1}{4} \langle T(\mathcal{R}^0, J\mathcal{R}^0), T(\mathcal{R}^0, e_l^0) \rangle, \\
 \langle \nabla_{\mathcal{R}^0}^{TB}(A(e_j^0)e_l^0), J\mathcal{R}^0 \rangle &= -\langle A(e_j^0)e_l^0, A(\mathcal{R}^0)J\mathcal{R}^0 \rangle = 0.
 \end{aligned}$$

By (3.40), (5.6a), (5.13), (5.53) and the fact that  $A$  exchanges  $TX_G$  and  $N_G$ , at  $x_0$ ,

$$\begin{aligned}
 (5.58) \quad \langle J\nabla_{\mathcal{R}}^{TX}e_k^H, \nabla_{\mathcal{R}}^{TX}e_l^{0,H} \rangle Z_k &= \langle J\nabla_{\mathcal{R}}^{TB}e_k, A(e_l^0)\mathcal{R} - \frac{1}{2}T(\mathcal{R}, e_l^0) \rangle Z_k \\
 &= \langle JA(\mathcal{R}^0)\mathcal{R}^0, -\frac{1}{2}T(\mathcal{R}, e_l^0) \rangle + 2\langle JA(\mathcal{R}^0)\mathcal{R}^\perp, A(e_l^0)\mathcal{R}^\perp \rangle \\
 &= \frac{1}{4} \langle T(\mathcal{R}^0, J\mathcal{R}^0), T(\mathcal{R}, e_l^0) \rangle + \frac{1}{2} \langle J\mathcal{R}^\perp, T(\mathcal{R}^0, e_j^0) \rangle \langle J\mathcal{R}^\perp, T(e_l^0, Je_j^0) \rangle.
 \end{aligned}$$

By (5.52), (5.56)-(5.58), at  $x_0$ ,

$$\begin{aligned}
 (5.59) \quad \left( \nabla \nabla \left\langle Je_k^H, e_l^{0,H} \right\rangle \right)_{(\mathcal{R}, \mathcal{R})} Z_k &= \frac{1}{3} \langle R^{TX_G}(\mathcal{R}^0, e_l^0)\mathcal{R}^0, J\mathcal{R}^0 \rangle \\
 &\quad + \langle 2\nabla_{\mathcal{R}^0}^{TX_G}(A(e_l^0)e_j^\perp)Z_j^\perp + R^{TB}(\mathcal{R}^\perp, e_l^0)\mathcal{R}^\perp + R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)e_l^0, J\mathcal{R}^0 \rangle \\
 &\quad - \langle 2R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)\mathcal{R}^\perp + R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)\mathcal{R}^0 + 3\nabla_{\mathcal{R}^0}^{TX_G}(A(e_i^0)e_j^\perp)Z_i^0Z_j^\perp, Je_l^0 \rangle \\
 &\quad - \frac{1}{2} \langle J\mathcal{R}^\perp, T(\mathcal{R}, A(e_l^0)\mathcal{R}) + \nabla_{\mathcal{R}}^{TY}(T(e_i, e_l^0))Z_i \rangle + \frac{1}{4} \langle T(\mathcal{R}^0, J\mathcal{R}^0), T(\mathcal{R}^\perp, e_l^0) \rangle \\
 &\quad - \frac{1}{4} \langle T(\mathcal{R}^\perp, J\mathcal{R}^0), T(\mathcal{R}, e_l^0) \rangle + \langle J\mathcal{R}^\perp, T(\mathcal{R}^0, e_j^0) \rangle \langle J\mathcal{R}^\perp, T(e_l^0, Je_j^0) \rangle.
 \end{aligned}$$

Observe that  $A(e_i^0)\mathcal{R}^0 \in N_G$ ,  $A(e_i^0)\mathcal{R}^\perp \in TX_G$ . By (5.5a), (5.5b), (5.5d) and (5.13),

$$\begin{aligned}
 (5.60) \quad \langle J\mathcal{R}^\perp, T(\mathcal{R}, A(e_l^0)\mathcal{R}) \rangle &= \langle J\mathcal{R}^\perp, T(\mathcal{R}, A(e_l^0)\mathcal{R}^0) \rangle + \langle J\mathcal{R}^\perp, T(\mathcal{R}, A(e_l^0)\mathcal{R}^\perp) \rangle \\
 &= \frac{1}{2} \langle JT(e_l^0, J\mathcal{R}^0), e_j^\perp \rangle \langle J\mathcal{R}^\perp, T(\mathcal{R}, e_j^\perp) \rangle + \langle J\mathcal{R}^\perp, T(\mathcal{R}, A(e_l^0)\mathcal{R}^\perp) \rangle \\
 &= -\frac{1}{2} \langle T(e_l^0, J\mathcal{R}^0), T(\mathcal{R}^0, \mathcal{R}^\perp) \rangle + \frac{1}{2} \langle J\mathcal{R}^\perp, T(\mathcal{R}, e_j^0) \rangle \langle J\mathcal{R}^\perp, T(e_l^0, Je_j^0) \rangle \\
 &\quad + \frac{1}{2} \langle JT(e_l^0, J\mathcal{R}^0), e_j^\perp \rangle \langle J\mathcal{R}^\perp, T(\mathcal{R}^\perp, e_j^\perp) \rangle.
 \end{aligned}$$

From (5.47), at  $x_0$ ,

$$\begin{aligned}
 (5.61) \quad (\nabla \nabla \langle \tilde{\mu}, T(e_k, e_l) \rangle)_{(\mathcal{R}, \mathcal{R})} &= \langle (\nabla_{\mathcal{R}}^{TY} \nabla_{\mathcal{R}}^{TY} \tilde{\mu})_{(\mathcal{R}, \mathcal{R})}, T(e_k, e_l) \rangle + 2 \langle \nabla_{\mathcal{R}}^{TY} \tilde{\mu}, \nabla_{\mathcal{R}}^{TY}(T(e_k, e_l)) \rangle \\
 &= \langle T(\mathcal{R}^\perp, J\mathcal{R}^\perp), T(e_k, e_l) \rangle - 2 \langle \nabla_{\mathcal{R}}^{TY}(T(e_k, e_l)), J\mathcal{R}^\perp \rangle.
 \end{aligned}$$

Finally, by (5.4), (5.51), (5.59), (5.60) and (5.61), we get (5.45b).  $\square$



We now examine the coefficients in the expansion of terms involving the moment map  $\tilde{\mu}$ .

Set

$$\begin{aligned}
 (5.62) \quad \mathcal{O}_2'' = & -\frac{1}{3} \langle (\nabla^{\cdot TY} \dot{g}^{\cdot TY})_{(\mathcal{R}, \mathcal{R})} J\mathcal{R}^\perp, J\mathcal{R}^\perp \rangle + \frac{1}{6} \langle \nabla_{\mathcal{R}}^{TY} (T(e_j, J_{x_0} e_i^0)), J\mathcal{R}^\perp \rangle Z_j Z_i^0 \\
 & + \frac{1}{3} \langle \nabla_{\mathcal{R}^0}^{NG} (A(e_j^0) e_i^0) Z_j^0 Z_i^0 + R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0) \mathcal{R}^0, \mathcal{R}^\perp \rangle \\
 & - \frac{1}{12} \sum_l \langle T(\mathcal{R}, e_l), J\mathcal{R}^\perp \rangle^2 + \frac{1}{4} \langle J\mathcal{R}^\perp, T(\mathcal{R}^\perp, e_l^0) \rangle \langle J\mathcal{R}^\perp, T(\mathcal{R}^0, e_l^0) \rangle \\
 & + \frac{7}{12} |T(\mathcal{R}^\perp, J\mathcal{R}^\perp)|^2 + \frac{1}{3} \langle T(\mathcal{R}^0, J\mathcal{R}^\perp), T(\mathcal{R}^\perp, J\mathcal{R}^\perp) \rangle.
 \end{aligned}$$

**Lemma 5.6.** *For  $|t| \leq 1$ , we have*

$$\begin{aligned}
 (5.63) \quad & \left| \frac{1}{t} \tilde{\mu} \right|_{g^{TY}}^2(tZ) = |Z^\perp|^2 - t \langle T(\mathcal{R}^\perp, J\mathcal{R}^\perp), J\mathcal{R}^\perp \rangle + t^2 \mathcal{O}_2'' + \mathcal{O}(t^3), \\
 & \langle \tilde{\mu}, \tilde{\mu}^E \rangle_{g^{TY}}(tZ) = -t \langle J\mathcal{R}^\perp, \tilde{\mu}_{x_0}^E \rangle \\
 & + t^2 \left( \frac{1}{2} \langle T(\mathcal{R}^\perp, J\mathcal{R}^\perp), \tilde{\mu}_{x_0}^E \rangle - \langle J\mathcal{R}^\perp, \nabla_{\mathcal{R}}^{TY} \tilde{\mu}_{x_0}^E \rangle \right) + \mathcal{O}(t^3).
 \end{aligned}$$

*Proof.* By (3.36), (3.38), (3.39), (5.6a), (5.53),  $\mathbf{J} = J$  and  $\tilde{\mu} = 0$  on  $P$ , we get, at  $x_0$ ,

$$\begin{aligned}
 (5.64) \quad & (\nabla_{e_k^H}^{TY} \nabla_{e_j^H}^{TY} \nabla_{e_i^H}^{TY} \tilde{\mu})_{x_0} = -P^{TY} J \nabla_{e_k^H}^{TX} \nabla_{e_j^H}^{TX} e_i^H - \frac{1}{2} T(e_k^H, P^{THX} J \nabla_{e_j^H}^{TX} e_i^H) \\
 & - \frac{1}{2} \nabla_{e_k^H}^{TY} (T(e_j^H, P^{THX} J e_i^H)) - \frac{1}{2} (\nabla_{e_j^H}^{TY} \dot{g}_{e_i^H}^{TY}) (\nabla_{e_k^H}^{TY} \tilde{\mu}) \\
 & - \frac{1}{2} (\nabla_{e_k^H}^{TY} \dot{g}_{e_i^H}^{TY}) (\nabla_{e_j^H}^{TY} \tilde{\mu}) - \frac{1}{2} \dot{g}_{e_i^H}^{TY} (\nabla_{e_k^H}^{TY} \nabla_{e_j^H}^{TY} \tilde{\mu}).
 \end{aligned}$$

From (3.40), (5.47), (5.53), (5.54), (5.55) and (5.64), we have

$$\begin{aligned}
 (5.65) \quad & (\nabla^{\cdot TY} \nabla^{\cdot TY} \nabla^{\cdot TY} \tilde{\mu})_{(\mathcal{R}, \mathcal{R}, \mathcal{R})} := (\nabla_{e_k^H}^{TY} \nabla_{e_j^H}^{TY} \nabla_{e_i^H}^{TY} \tilde{\mu})_{x_0} Z_k Z_j Z_i \\
 & = -J \nabla_{\mathcal{R}^0}^{NG} (A(e_j^0) e_i^0) Z_j^0 Z_i^0 - 3JA(\mathcal{R}^0) A(\mathcal{R}^0) \mathcal{R}^\perp - 2P^{TY} J R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0) \mathcal{R}^\perp \\
 & \quad - P^{TY} J R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0) \mathcal{R}^0 - T(\mathcal{R}, JA(\mathcal{R}^0) \mathcal{R}^\perp) \\
 & \quad - \frac{1}{2} \nabla_{\mathcal{R}}^{TY} (T(e_j^H, P^{THX} J e_i^H)) Z_j Z_i + (\nabla^{\cdot TY} \dot{g}^{\cdot TY})_{(\mathcal{R}, \mathcal{R})} J\mathcal{R}^\perp - \frac{1}{2} \dot{g}_{\mathcal{R}}^{TY} (T(\mathcal{R}^\perp, J\mathcal{R}^\perp)).
 \end{aligned}$$

Now by (3.50), (5.47), (5.65), and  $\tilde{\mu} = 0$  on  $P$ , we have

$$\begin{aligned}
 (5.66) \quad & \left| \frac{1}{t} \tilde{\mu} \right|_{g^{TY}}^2(tZ) = \sum_{k=2}^4 \frac{1}{k!} \frac{\partial^k}{\partial t^k} \left( |\tilde{\mu}|_{g^{TY}}^2(tZ) \right) \Big|_{t=0} t^{k-2} + \mathcal{O}(t^3) \\
 & = |\nabla_{\mathcal{R}}^{TY} \tilde{\mu}|_{x_0}^2 + t \langle (\nabla^{\cdot TY} \nabla^{\cdot TY} \tilde{\mu})_{(\mathcal{R}, \mathcal{R})}, \nabla_{\mathcal{R}}^{TY} \tilde{\mu} \rangle_{x_0} \\
 & \quad + \frac{t^2}{4!} \left( 8 \langle (\nabla^{\cdot TY} \nabla^{\cdot TY} \nabla^{\cdot TY} \tilde{\mu})_{(\mathcal{R}, \mathcal{R}, \mathcal{R})}, \nabla_{\mathcal{R}}^{TY} \tilde{\mu} \rangle_{x_0} + 6 |(\nabla^{\cdot TY} \nabla^{\cdot TY} \tilde{\mu})_{(\mathcal{R}, \mathcal{R})}|_{x_0}^2 \right) + \mathcal{O}(t^3).
 \end{aligned}$$

By (5.5c),

$$(5.67) \quad T(\mathcal{R}^0, J\mathcal{R}^\perp) = \frac{1}{2}T(\mathcal{R}^\perp, J\mathcal{R}^0).$$

From (1.6), (5.13), (5.47), (5.65), (5.66) and (5.67), we get the coefficients of  $t^0, t^1$  in the expansion of  $|\frac{1}{t}\tilde{\mu}|_{g_{TY}}^2(tZ)$  in (5.63), and the coefficients of  $t^2$  is

$$(5.68) \quad \begin{aligned} & \frac{1}{3} \langle J\nabla_{\mathcal{R}^0}^{NG} (A(e_j^0)e_i^0)Z_j^0Z_i^0 + 3JA(\mathcal{R}^0)A(\mathcal{R}^0)\mathcal{R}^\perp + JR^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)\mathcal{R}^0, J\mathcal{R}^\perp \rangle \\ & + \frac{1}{3} \langle 2JR^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)\mathcal{R}^\perp + T(\mathcal{R}, JA(\mathcal{R}^0)\mathcal{R}^\perp), J\mathcal{R}^\perp \rangle \\ & - \frac{1}{3} \langle (\nabla_{\mathcal{R}}^{TY} \dot{g}^{TY})_{(\mathcal{R}, \mathcal{R})} J\mathcal{R}^\perp, J\mathcal{R}^\perp \rangle + \frac{1}{6} \langle \nabla_{\mathcal{R}}^{TY} (T(e_j^H, P^{THX} J e_i^H)) Z_j Z_i, J\mathcal{R}^\perp \rangle \\ & + \frac{1}{3} \langle T(\mathcal{R}, J\mathcal{R}^\perp), T(\mathcal{R}^\perp, J\mathcal{R}^\perp) \rangle + \frac{1}{4} |T(\mathcal{R}^\perp, J\mathcal{R}^\perp)|^2 \\ & = -\frac{1}{3} \langle (\nabla_{\mathcal{R}}^{TY} \dot{g}^{TY})_{(\mathcal{R}, \mathcal{R})} J\mathcal{R}^\perp, J\mathcal{R}^\perp \rangle + \frac{1}{6} \langle \nabla_{\mathcal{R}}^{TY} (T(e_j^H, P^{THX} J e_i^H)) Z_j Z_i, J\mathcal{R}^\perp \rangle \\ & + \frac{1}{3} \langle \nabla_{\mathcal{R}^0}^{NG} (A(e_j^0)e_i^0)Z_j^0Z_i^0 + R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0)\mathcal{R}^0, \mathcal{R}^\perp \rangle \\ & - \frac{1}{4} \sum_j \langle T(\mathcal{R}^0, e_j^0), J\mathcal{R}^\perp \rangle^2 + \frac{1}{6} \langle T(\mathcal{R}, e_j^0), J\mathcal{R}^\perp \rangle \langle T(\mathcal{R}^0, e_j^0), J\mathcal{R}^\perp \rangle \\ & + \frac{7}{12} |T(\mathcal{R}^\perp, J\mathcal{R}^\perp)|^2 + \frac{1}{3} \langle T(\mathcal{R}^0, J\mathcal{R}^\perp), T(\mathcal{R}^\perp, J\mathcal{R}^\perp) \rangle. \end{aligned}$$

To get (5.63) from (5.68), we need to compute  $\nabla_{e_k^H}^{TY} (T(e_j^H, P^{THX} J e_i^H))$ .

For  $W$  a section of  $TX$ ,  $U$  a section of  $TB$ , we have

$$(5.69) \quad \begin{aligned} \langle \nabla_{e_k^H}^{THX} P^{THX} W, U^H \rangle &= e_k^H \langle W, U^H \rangle - \langle P^{THX} W, \nabla_{e_k^H}^{TX} U^H \rangle \\ &= \langle P^{THX} \nabla_{e_k^H}^{TX} W, U^H \rangle + \langle P^{TY} W, \nabla_{e_k^H}^{TX} U^H \rangle. \end{aligned}$$

From (1.7), (5.69), we get at  $x_0$ ,

$$(5.70) \quad \nabla_{e_k^H}^{THX} P^{THX} W = P^{THX} \nabla_{e_k^H}^{TX} W - \frac{1}{2} \langle T(e_k^H, e_l^H), P^{TY} W \rangle e_l^H.$$

Remark that  $J e_i^{\perp, H} \in TY$ ,  $J e_i^0 \in T^H X$  only hold on  $P$ .

From (3.40), (5.5b), (5.6a), (5.13), (5.31) and (5.70),

$$(5.71) \quad \begin{aligned} (\nabla_{e_k^H}^{THX} P^{THX} J e_i^{\perp, H})_{x_0} &= JA(P^{TXG} e_k) e_i^\perp - \frac{1}{2} JT(e_k, e_i^\perp) - \frac{1}{2} \langle T(e_k, e_l), J e_i^\perp \rangle e_l \\ &= -\frac{1}{2} JT(e_k, e_i^\perp) - \frac{1}{2} \langle T(e_k, e_l) - T(P^{TXG} e_k, P^{TXG} e_l), J e_i^\perp \rangle e_l, \\ (\nabla_{e_k}^{THX} P^{THX} J e_i^0)_{x_0} &= P^{THX} J \nabla_{e_k^H}^{TX} e_i^{0, H} = JA(e_i^0) P^{NG} e_k - \frac{1}{2} JT(e_k, e_i^0) \\ &= -\frac{1}{2} JT(e_k, e_i^0) + \frac{1}{2} \langle J P^{NG} e_k, T(e_i^0, e_l^0) \rangle e_l^0, \\ (\nabla_{e_k}^{TB} J_{x_0} e_i^0)_{x_0} &= A(J_{x_0} e_i^0) e_k = -\frac{1}{2} JT(P^{TXG} e_k, e_i^0) + \frac{1}{2} \langle J P^{NG} e_k, T(e_i^0, e_l^0) \rangle e_l^0. \end{aligned}$$

From (5.71), we get at  $x_0$  that

$$\begin{aligned}
(5.72) \quad & \left\langle \nabla_{\mathcal{R}}^{TY} (T(e_j^H, P^{TX} J e_i^H)) Z_j Z_i, J\mathcal{R}^\perp \right\rangle - \left\langle \nabla_{\mathcal{R}}^{TY} (T(e_j, J_{x_0} e_i^0)) Z_j Z_i^0, J\mathcal{R}^\perp \right\rangle \\
&= \left\langle T(e_j, \nabla_{\mathcal{R}}^{TX} P^{TX} J e_i^H - \nabla_{\mathcal{R}}^{TX} (J_{x_0} P^{TX} e_i)^H) Z_j Z_i, J\mathcal{R}^\perp \right\rangle \\
&= \left\langle T \left( \mathcal{R}, -\frac{1}{2} J T(\mathcal{R}, \mathcal{R}^\perp) - \frac{1}{2} \langle T(\mathcal{R}, e_l) - T(\mathcal{R}^0, P^{TX_G} e_l), J\mathcal{R}^\perp \rangle e_l \right), J\mathcal{R}^\perp \right\rangle \\
&\quad + \left\langle T \left( e_j, -\frac{1}{2} J T(e_k, e_i^0) + \frac{1}{2} J T(P^{TX_G} e_k, e_i^0) \right) Z_k Z_j Z_i^0, J\mathcal{R}^\perp \right\rangle \\
&= -\frac{1}{2} \sum_l \langle T(\mathcal{R}, e_l), J\mathcal{R}^\perp \rangle^2 + \frac{1}{2} \langle T(\mathcal{R}, e_l^0), J\mathcal{R}^\perp \rangle \langle T(\mathcal{R}^0, e_l^0), J\mathcal{R}^\perp \rangle.
\end{aligned}$$

From (5.68) and (5.72),  $\mathcal{O}_2''$  is the coefficient of  $t^2$  in the expansion of  $|\frac{1}{t} \tilde{\mu}|_{g^{TY}}^2(tZ)$ .

By (5.47), we get also the second equation of (5.63).

The proof of Lemma 5.6 is completed.  $\square$

The following is the main result of this Subsection.

**Theorem 5.7.** *The following identities hold,*

$$\begin{aligned}
(5.73) \quad & \mathcal{O}_1 = 2\pi\sqrt{-1} \langle JT(\mathcal{R}^\perp, e_i^0), \mathcal{R}^\perp \rangle \nabla_{0, e_i^0} + 2\pi\sqrt{-1} \langle JT(\mathcal{R}, e_i^\perp), \mathcal{R}^\perp \rangle \nabla_{0, e_i^\perp} \\
&\quad + \pi\sqrt{-1} \langle JT(\mathcal{R}^0, e_i^\perp), e_i^\perp \rangle - \langle JT(e_i^0, J e_j^0), \mathcal{R}^\perp \rangle \nabla_{0, e_i^0} \nabla_{0, e_j^0} \\
&\quad + 4\pi^2 \langle JT(\mathcal{R}^\perp, J\mathcal{R}^\perp), \mathcal{R}^\perp \rangle + 4\pi\sqrt{-1} \langle J\mathcal{R}^\perp, \tilde{\mu}_{x_0}^E \rangle, \\
& \mathcal{O}_2 = \mathcal{O}_2' + 4\pi^2 \mathcal{O}_2'' - 4\pi\sqrt{-1} \left( \frac{1}{2} \langle T(\mathcal{R}^\perp, J\mathcal{R}^\perp), \tilde{\mu}_{x_0}^E \rangle - \langle J\mathcal{R}^\perp, \nabla_{\mathcal{R}}^{TY} \tilde{\mu}^E \rangle \right) \\
&\quad - \langle \tilde{\mu}_{x_0}^E, \tilde{\mu}_{x_0}^E \rangle_{g^{TY}}.
\end{aligned}$$

*Proof.* By (5.5e),

$$(5.74) \quad \langle JT(\mathcal{R}, e_i), e_i \rangle = \langle JT(\mathcal{R}^0, e_i^\perp), e_i^\perp \rangle.$$

By (5.45a), (5.50) and (5.74),

$$\begin{aligned}
(5.75) \quad & -\frac{2}{3} (\partial_{\mathcal{R}} R^{LB})_{x_0}(\mathcal{R}, e_i) \nabla_{0, e_i} \\
&= 2\pi\sqrt{-1} \left( \langle JT(\mathcal{R}^\perp, e_i^0), \mathcal{R}^\perp \rangle \nabla_{0, e_i^0} + \langle JT(\mathcal{R}, e_i^\perp), \mathcal{R}^\perp \rangle \nabla_{0, e_i^\perp} \right), \\
&\quad -\frac{1}{3} (\partial_i R^{LB})_{x_0}(\mathcal{R}, e_i) = \pi\sqrt{-1} \langle JT(\mathcal{R}^0, e_i^\perp), e_i^\perp \rangle.
\end{aligned}$$

From (5.22), (5.26), (5.63) and (5.75), we get (5.73).  $\square$

**5.3. Computation of the coefficient  $\Phi_1$ .** Recall that the operator  $\mathcal{L}_2^0$  is defined in (5.22),  $P_{\mathcal{L}^\perp}$  is the orthogonal projection from  $L^2(\mathbb{R}^{n_0})$  onto  $\text{Ker } \mathcal{L}^\perp$  and  $P_{\mathcal{L}}$  is the orthogonal projection from  $L^2(\mathbb{R}^{2n-2n_0})$  onto  $\text{Ker } \mathcal{L}$  as in (3.19).

For  $Z^\perp \in \mathbb{R}^{n_0}$ , set

$$\begin{aligned}
 \Psi_{1,1}(Z^\perp) &= \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{O}_1 (\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{O}_1 P^N \right) ((0, Z^\perp), (0, Z^\perp)), \\
 \Psi_{1,2}(Z^\perp) &= - \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{O}_2 P^N \right) ((0, Z^\perp), (0, Z^\perp)), \\
 \Psi_{1,3}(Z^\perp) &= \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{O}_1 P^N \mathcal{O}_1 (\mathcal{L}_2^0)^{-1} P^{N^\perp} \right) ((0, Z^\perp), (0, Z^\perp)), \\
 \Psi_{1,4}(Z^\perp) &= \left( P^N \mathcal{O}_1 (\mathcal{L}_2^0)^{-2} P^{N^\perp} \mathcal{O}_1 P^N \right) ((0, Z^\perp), (0, Z^\perp)), \\
 \tilde{\Psi}_{1,1}(Z^\perp) &= \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} P_{\mathcal{L}^\perp} \mathcal{O}_1 (\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{O}_1 P^N \right) ((0, Z^\perp), (0, Z^\perp)), \\
 \tilde{\Psi}_{1,2}(Z^\perp) &= - \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} P_{\mathcal{L}^\perp} \mathcal{O}_2 P^N \right) ((0, Z^\perp), (0, Z^\perp)), \\
 \Phi_{1,i} &= \int_{\mathbb{R}^{n_0}} \Psi_{1,i}(Z^\perp) dv_{N_G}(Z^\perp), \quad \text{for } i = 1, 2, 3, 4.
 \end{aligned}
 \tag{5.76}$$

**Proposition 5.8.** *The following two identities hold for  $i = 1, 2$ ,*

$$\int_{\mathbb{R}^{n_0}} \tilde{\Psi}_{1,i}(Z^\perp) dv_{N_G}(Z^\perp) = \Phi_{1,i}.
 \tag{5.77}$$

*Proof.* In fact, in our case, by (3.21),  $P^N = P_{\mathcal{L}} \otimes P_{\mathcal{L}^\perp} \otimes \text{Id}_E$ .

By (3.18), (3.19),

$$\left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{O}_2 P^N \right) (Z, (0, Z'^\perp)) = \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{O}_2 P_{\mathcal{L}}(\cdot, 0) G^\perp \right) (Z) G^\perp(Z'^\perp).
 \tag{5.78}$$

From Theorem 3.1, (5.78),

$$\begin{aligned}
 \Phi_{1,2} &= \left\langle \left( -(\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{O}_2 P_{\mathcal{L}}(\cdot, 0) G^\perp \right) (0, Z^\perp), G^\perp(Z^\perp) \right\rangle_{L^2(\mathbb{R}^{n_0})} \\
 &= \left\langle \left( -(\mathcal{L}_2^0)^{-1} P^{N^\perp} P_{\mathcal{L}^\perp} \mathcal{O}_2 P_{\mathcal{L}}(\cdot, 0) G^\perp \right) (0, Z^\perp), G^\perp(Z^\perp) \right\rangle_{L^2(\mathbb{R}^{n_0})} \\
 &= \int_{\mathbb{R}^{n_0}} \tilde{\Psi}_{1,2}(Z^\perp) dv_{N_G}(Z^\perp).
 \end{aligned}
 \tag{5.79}$$

In the same way, we get (5.77) for  $i = 1$ . □

Note that the restriction of  $\|\cdot\|_{t,0}$  in (2.115) on  $\mathcal{C}^\infty(\mathbb{R}^{2n-n_0}, E_{G,x_0})$  does not depend on  $t$  and we denote it by  $\|\cdot\|_0$ .

Since  $\mathcal{L}_2^t$  in (5.22) is a self-adjoint elliptic operator with respect to  $\|\cdot\|_0$  as we conjugated the operator with  $\kappa^{1/2}$ ,  $\mathcal{L}_2^0$  and  $\mathcal{O}_r$  are also formally self-adjoint with respect to  $\|\cdot\|_0$ . Thus in the right hand side of (3.62), the third and fourth terms are the adjoints of the first two terms.

From (3.62), (5.1) and (5.76), we get

$$\Phi_1 = \Phi_{1,1} + \Phi_{1,2} + (\Phi_{1,1} + \Phi_{1,2})^* + \Phi_{1,3} - \Phi_{1,4}.
 \tag{5.80}$$

From (5.76), (5.77), (5.80), we learn that in order to compute  $\Phi_1$ , we only need to evaluate  $\Psi_{1,i}$  and  $\tilde{\Psi}_{1,i}$  ( $i \in \{1, 2, 3, 4\}$ ).

**Lemma 5.9.** *The following identity holds,*

$$(5.81) \quad \tilde{\Psi}_{1,1}(Z^\perp) = -\frac{1}{8\pi} \left| T\left(\frac{\partial}{\partial \bar{z}_j}, e_k^\perp\right) \right|^2 P_{\mathcal{L}^\perp}(Z^\perp, Z^\perp).$$

*Proof.* Recall that the operators  $b_i$ ,  $b_i^\perp$ ,  $b_j^\perp$  and  $b_j^{\perp+}$  have been defined in (3.8). In particular, by (5.15), one has

$$(5.82) \quad 4\pi Z_j^\perp = b_j^\perp + b_j^{\perp+}, \quad \nabla_{0, e_j^\perp} = \frac{\partial}{\partial Z_j^\perp} = \frac{1}{2}(b_j^{\perp+} - b_j^\perp).$$

By (3.8), (3.9) and (5.82), set

$$(5.83) \quad \begin{aligned} B_{jk}^\perp &= (4\pi)^2 Z_j^\perp Z_k^\perp = b_j^{\perp+} b_k^{\perp+} + b_k^\perp b_j^{\perp+} + b_j^\perp b_k^{\perp+} + b_j^{\perp+} b_k^\perp + 4\pi \delta_{jk}, \\ B_{ijk}^\perp &= b_i^\perp b_j^\perp b_k^\perp + 3b_i^\perp b_j^\perp b_k^{\perp+} + 3b_i^\perp b_j^{\perp+} b_k^{\perp+} + b_i^{\perp+} b_j^{\perp+} b_k^{\perp+}. \end{aligned}$$

If  $a_{ijk}$  is symmetric on  $i, j, k$ , then by (3.8), (3.9), (5.82) and (5.83), one verifies

$$(5.84) \quad a_{ijk}(4\pi)^3 Z_i^\perp Z_j^\perp Z_k^\perp = a_{ijk} B_{ijk}^\perp + 12\pi a_{ijj}(b_i^\perp + b_i^{\perp+}).$$

By (3.9), (5.5e), (5.14), (5.82), (5.83) and the fact that  $T(\cdot, \cdot)$  is anti-symmetric, we get

$$(5.85) \quad \begin{aligned} 2\pi \langle JT(\mathcal{R}^\perp, e_i^\perp), \mathcal{R}^\perp \rangle \nabla_{0, e_i^\perp} &= \frac{1}{16\pi} \tilde{\mathcal{T}}_{jik} B_{jk}^\perp (b_i^{\perp+} - b_i^\perp) \\ &= \frac{1}{16\pi} \tilde{\mathcal{T}}_{jik} [(b_j^\perp b_k^{\perp+} + b_j^{\perp+} b_k^\perp) b_i^{\perp+} - (b_j^{\perp+} b_k^{\perp+} + b_k^\perp b_j^{\perp+} + b_j^\perp b_k^{\perp+}) b_i^\perp] \\ &= -\frac{1}{8\pi} \tilde{\mathcal{T}}_{ijk} (b_j^\perp b_k^{\perp+} + b_j^{\perp+} b_k^\perp) b_i^{\perp+}. \end{aligned}$$

By Theorem 5.1, Remark 5.2, (3.9), (3.12), (5.14), (5.73), (5.83)-(5.85), we can reformulate  $\mathcal{O}_1$  as follows by using the creation and annihilation operators introduced in (3.8),

$$(5.86) \quad \begin{aligned} \mathcal{O}_1 &= -\frac{\sqrt{-1}}{8\pi} \left\langle JT\left(\frac{\partial}{\partial \bar{z}_i}, e_j^\perp\right), e_k^\perp \right\rangle B_{jk}^\perp b_i^+ + b_i \frac{\sqrt{-1}}{8\pi} \left\langle JT\left(\frac{\partial}{\partial \bar{z}_i}, e_j^\perp\right), e_k^\perp \right\rangle B_{jk}^\perp \\ &\quad + \frac{\sqrt{-1}}{4} \left\langle JT(\mathcal{R}^0, e_i^\perp), e_j^\perp \right\rangle (b_i^{\perp+} b_j^{\perp+} - b_i^\perp b_j^\perp) - \frac{\sqrt{-1}}{8\pi} \tilde{\mathcal{T}}_{ijk} (b_j^\perp b_k^{\perp+} + b_j^{\perp+} b_k^\perp) b_i^{\perp+} \\ &\quad - \frac{\sqrt{-1}}{4\pi} \left\langle JT\left(\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial \bar{z}_j}\right), e_k^\perp \right\rangle (b_k^{\perp+} + b_k^\perp) (2b_j b_i^+ + 4\pi \delta_{ij}) + \sqrt{-1} \langle J e_j^\perp, \tilde{\mu}_{x_0}^E \rangle (b_j^{\perp+} + b_j^\perp) \\ &\quad + \frac{1}{16\pi} \left\langle JT(e_i^\perp, J e_j^\perp), e_k^\perp \right\rangle [B_{ijk}^\perp + 12\pi \delta_{ik} (b_j^{\perp+} + b_j^\perp)] \\ &= -\frac{\sqrt{-1}}{8\pi} \mathcal{T}_{jk} \left(\frac{\partial}{\partial \bar{z}_i}\right) B_{jk}^\perp b_i^+ + \frac{\sqrt{-1}}{8\pi} \mathcal{T}_{jk} \left(\frac{\partial}{\partial \bar{z}_i}\right) b_i B_{jk}^\perp + \frac{\sqrt{-1}}{4} \mathcal{T}_{ij}(\mathcal{R}^0) (b_i^{\perp+} b_j^{\perp+} - b_i^\perp b_j^\perp) \\ &\quad + \sqrt{-1} \langle J e_j^\perp, \tilde{\mu}_{x_0}^E \rangle (b_j^{\perp+} + b_j^\perp) - \frac{\sqrt{-1}}{4\pi} \left\langle JT\left(\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial \bar{z}_j}\right), e_k^\perp \right\rangle (b_k^{\perp+} + b_k^\perp) (2b_j b_i^+ + 4\pi \delta_{ij}) \\ &\quad - \frac{\sqrt{-1}}{8\pi} \tilde{\mathcal{T}}_{ijk} (b_j^\perp b_k^{\perp+} + b_j^{\perp+} b_k^\perp) b_i^{\perp+} + \frac{1}{16\pi} \mathcal{T}_{ijk} [B_{ijk}^\perp + 12\pi \delta_{ik} (b_j^{\perp+} + b_j^\perp)]. \end{aligned}$$

From Theorem 3.1, (3.54), (5.83), (5.86) and  $a_i = a_i^+ = 2\pi$ , we get

$$(5.87) \quad ((\mathcal{L}_2^0)^{-1} \mathcal{O}_1 P^N)(Z, Z') = \sqrt{-1} \left\{ \frac{b_l}{8\pi} \mathcal{T}_{kk} \left( \frac{\partial}{\partial \bar{z}_l^0} \right) + \langle J e_k^\perp, \tilde{\mu}_{x_0}^E \rangle \frac{b_k^\perp}{4\pi} \right. \\ \left. - \left\langle JT \left( \frac{\partial}{\partial \bar{z}_l^0}, \frac{\partial}{\partial \bar{z}_l^0} \right), e_k^\perp \right\rangle \frac{b_k^\perp}{4\pi} - \frac{b_l^\perp b_k^\perp}{32\pi} \mathcal{T}_{kl}(z^0 + \bar{z}'^0) \right. \\ \left. - \frac{\sqrt{-1}}{16\pi} \mathcal{T}_{klm} \left[ \frac{b_m^\perp b_l^\perp b_k^\perp}{12\pi} + 3b_k^\perp \delta_{lm} \right] \right\} P^N(Z, Z').$$

By Theorem 3.1, (3.55), (5.83) and (5.86),

$$(5.88) \quad P^{N^\perp} P_{\mathcal{L}^\perp} \mathcal{O}_1 = \sqrt{-1} P^{N^\perp} P_{\mathcal{L}^\perp} \left\{ -\frac{1}{2} \mathcal{T}_{jj} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) b_i^+ + \frac{1}{2} \mathcal{T}_{jj} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) b_i + \langle J e_j^\perp, \tilde{\mu}_{x_0}^E \rangle b_j^{\perp+} \right. \\ \left. - \frac{1}{4\pi} \left\langle JT \left( \frac{\partial}{\partial \bar{z}_i^0}, \frac{\partial}{\partial \bar{z}_j^0} \right), e_{j'}^\perp \right\rangle b_{j'}^{\perp+} (2b_j b_i^+ + 4\pi \delta_{ij}) \right. \\ \left. + \frac{1}{4} \left( \mathcal{T}_{jj'}(\mathcal{R}^0) - \mathcal{T}_{jj'} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) \frac{b_i^+}{2\pi} + \mathcal{T}_{jj'} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) \frac{b_i}{2\pi} \right) b_j^{\perp+} b_{j'}^{\perp+} \right. \\ \left. - \frac{\sqrt{-1}}{16\pi} \mathcal{T}_{ijj'} [b_i^{\perp+} b_j^{\perp+} b_{j'}^{\perp+} + 12\pi \delta_{ij'} b_j^{\perp+}] \right\}.$$

In the following equation, by (3.9), (3.54), (3.55), we only need to pair the terms in (5.87) and (5.88) which have the same length on  $b_j^{\perp+}$  and  $b_j^\perp$ , and the total degree on  $b_i, b_i^+, z^0, \bar{z}^0$  should not be zero. Thus by (3.9), (3.54), (5.87) and (5.88),

$$(5.89) \quad \left( P^{N^\perp} P_{\mathcal{L}^\perp} \mathcal{O}_1 (\mathcal{L}_2^0)^{-1} \mathcal{O}_1 P^N \right) (Z, (0, Z'^\perp)) = \left\{ P^{N^\perp} \left[ -\frac{1}{16\pi} \left( \sum_{ij} b_i \mathcal{T}_{jj} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) \right)^2 \right. \right. \\ \left. \left. + \frac{1}{128\pi} \left( \mathcal{T}_{jj'}(\mathcal{R}^0) + \frac{b_i}{2\pi} \mathcal{T}_{jj'} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) \right) b_j^{\perp+} b_{j'}^{\perp+} \cdot b_l^\perp b_k^\perp \mathcal{T}_{kl}(z^0) \right] P^N \right\} (Z, (0, Z'^\perp)).$$

From (3.9), (3.54), (5.5d), (5.14), (5.89) and  $a_i = a_i^+ = 2\pi$ , one gets

$$(5.90) \quad \left( P^{N^\perp} P_{\mathcal{L}^\perp} \mathcal{O}_1 (\mathcal{L}_2^0)^{-1} \mathcal{O}_1 P^N \right) (Z, (0, Z'^\perp)) = \left\{ P^{N^\perp} \left[ -\frac{1}{16\pi} \left( \sum_{ij} b_i \mathcal{T}_{jj} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) \right)^2 \right. \right. \\ \left. \left. + \frac{1}{8} \left\langle 2\pi JT(\mathcal{R}^0, e_l^\perp) + b_i JT \left( \frac{\partial}{\partial \bar{z}_i^0}, e_l^\perp \right), JT(z^0, e_l^\perp) \right\rangle \right] P^N \right\} (Z, (0, Z'^\perp)).$$

Set  $P_{\mathcal{L}}^\perp = \text{Id}_{L^2(\mathbb{R}^{2n-2n_0})} - P_{\mathcal{L}}$ .

Let  $h_i(Z^0)$  (resp.  $F(Z^0)$ ) be polynomials in  $Z^0$  with degree 1 (resp. 2) and  $a_{ij} \in \mathbb{C}$ .

By Theorem 3.1, (3.9) and (3.54),

$$(5.91) \quad (F(Z^0) P_{\mathcal{L}})(Z^0, 0) = \left( \frac{1}{2} \frac{\partial^2 F}{\partial z_i^0 \partial \bar{z}_j^0} z_i^0 z_j^0 + \frac{\partial^2 F}{\partial z_i^0 \partial \bar{z}_j^0} z_i^0 \frac{b_j}{a_j} + \frac{1}{2} \frac{\partial^2 F}{\partial \bar{z}_i^0 \partial \bar{z}_j^0} \frac{b_i b_j}{a_i a_j} \right) P_{\mathcal{L}}(Z^0, 0).$$

By Theorem 3.1, (3.8), (3.9), (3.19), (3.54), (5.91) and  $a_j = 2\pi$ , we have

$$\begin{aligned}
 (P_{\mathcal{L}}^{\perp} F P_{\mathcal{L}})(0, 0) &= -\frac{1}{\pi} \frac{\partial^2 F}{\partial z_i^0 \partial \bar{z}_i^0}, \\
 (\mathcal{L}^{-1} P_{\mathcal{L}}^{\perp} a_{ij} b_i b_j P_{\mathcal{L}})(0, 0) &= (\mathcal{L}^{-1} P_{\mathcal{L}}^{\perp} h_i P_{\mathcal{L}})(0, 0) = 0, \\
 (\mathcal{L}^{-1} P_{\mathcal{L}}^{\perp} h_i b_i P_{\mathcal{L}})(0, 0) &= (\mathcal{L}^{-1} P_{\mathcal{L}}^{\perp} b_i h_i P_{\mathcal{L}})(0, 0) = -\frac{1}{2\pi} \frac{\partial h_i}{\partial z_i^0}, \\
 (\mathcal{L}^{-1} P_{\mathcal{L}}^{\perp} F P_{\mathcal{L}})(0, 0) &= -\frac{1}{4\pi^2} \frac{\partial^2 F}{\partial z_i^0 \partial \bar{z}_i^0}, \\
 (\mathcal{L}^{-1} P_{\mathcal{L}}^{\perp} b_i F b_j P_{\mathcal{L}})(0, 0) &= -(\mathcal{L}^{-1} P_{\mathcal{L}}^{\perp} b_i b_j F P_{\mathcal{L}})(0, 0) = -\frac{1}{2\pi} \frac{\partial^2 F}{\partial z_i^0 \partial z_j^0}, \\
 (\mathcal{L}^{-1} P_{\mathcal{L}}^{\perp} F b_i b_j P_{\mathcal{L}})(0, 0) &= -\frac{3}{2\pi} \frac{\partial^2 F}{\partial z_i^0 \partial z_j^0}, \\
 \left( \mathcal{L}^{-1} P_{\mathcal{L}}^{\perp} \left( \sum_i b_i h_i \right)^2 P_{\mathcal{L}} \right)(0, 0) &= -\frac{1}{2\pi} \left( \frac{\partial h_i}{\partial z_j^0} \frac{\partial h_j}{\partial z_i^0} - \left( \sum_i \frac{\partial h_i}{\partial z_i^0} \right)^2 \right).
 \end{aligned}
 \tag{5.92}$$

Finally by (5.77), (5.90), (5.92) and  $\mathcal{L}_2^0 = \mathcal{L} + \mathcal{L}^{\perp}$ , we get (5.81).  $\square$

**Lemma 5.10.** *The following identity holds,*

$$(5.93) \quad \Phi_{1,3} = \Phi_{1,4}.$$

*Proof.* Let  $\mathcal{F}_2 \in T_{x_0}^* X_G$  with values in real polynomials on  $Z^{\perp}$  with even degree,  $\mathcal{F}_1 \in N_{G,x_0}^*$ ,  $\mathcal{F}_3(Z^{\perp})$  a polynomial on  $Z^{\perp}$  with odd degree, be defined by

$$\begin{aligned}
 \mathcal{F}_1(e_k^{\perp}) &= \sqrt{-1} \langle J e_k^{\perp}, \tilde{\mu}_{x_0}^E \rangle - \sqrt{-1} \left\langle J T\left(\frac{\partial}{\partial z_l^0}, \frac{\partial}{\partial \bar{z}_l^0}\right), e_k^{\perp} \right\rangle + \frac{3}{4} \mathcal{T}_{lk}, \\
 \mathcal{F}_2(\cdot, Z^{\perp}) P^N(Z, Z') &= \left( \mathcal{T}_{kl}(\cdot) \frac{b_l^{\perp} b_k^{\perp}}{32\pi} P^N \right)(Z, Z'), \\
 \mathcal{F}_3(Z^{\perp}) P^N(Z, Z') &= \frac{1}{16\pi} \left( \mathcal{T}_{klm} \frac{b_m^{\perp} b_l^{\perp} b_k^{\perp}}{12\pi} P^N \right)(Z, Z').
 \end{aligned}
 \tag{5.94}$$

Then from (3.54), (5.87) and (5.94),

$$\begin{aligned}
 (5.95) \quad ((\mathcal{L}_2^0)^{-1} \mathcal{O}_1 P^N)(Z, Z') &= \left( \frac{\sqrt{-1}}{4} \mathcal{T}_{kk}(\bar{z}^0 - \bar{z}'^0) - \sqrt{-1} \mathcal{F}_2(z^0 + \bar{z}'^0, Z^{\perp}) \right. \\
 &\quad \left. + (\mathcal{F}_1 + \mathcal{F}_3)(Z^{\perp}) \right) P^N(Z, Z').
 \end{aligned}$$

Observe that  $\mathcal{F}_i(Z^{\perp})^* = \mathcal{F}_i(Z^{\perp})$  for  $i = 1, 3$ , thus from (5.87) and (5.95),

$$\begin{aligned}
 (5.96) \quad (P^N \mathcal{O}_1 (\mathcal{L}_2^0)^{-1})(Z', Z) &= (((\mathcal{L}_2^0)^{-1} \mathcal{O}_1 P^N)(Z, Z'))^* \\
 &= \left( -\frac{\sqrt{-1}}{4} \mathcal{T}_{kk}(z^0 - z'^0) + \sqrt{-1} \mathcal{F}_2(\bar{z}^0 + z'^0, Z^{\perp}) + (\mathcal{F}_1 + \mathcal{F}_3)(Z^{\perp}) \right) P^N(Z', Z).
 \end{aligned}$$

For  $h_1(z^0), h_2(\bar{z}^0)$  two linear functions on  $z^0, \bar{z}^0$ , by Theorem 3.1, (3.54),

$$(5.97) \quad (P_{\mathcal{L}} h_1(z^0) h_2(\bar{z}^0) P_{\mathcal{L}})(0, 0) = \left( P_{\mathcal{L}} h_1(z^0) \frac{\partial h_2}{\partial \bar{z}_i^0} \frac{b_i}{2\pi} P_{\mathcal{L}} \right)(0, 0) = \frac{1}{\pi} \frac{\partial h_1}{\partial z_i^0} \frac{\partial h_2}{\partial \bar{z}_i^0}.$$

From (5.76), (5.95)-(5.97),

$$(5.98) \quad \Psi_{1,3}(Z^\perp) = \left[ \left( (\mathcal{F}_1 + \mathcal{F}_3)(Z^\perp) \right)^2 + \frac{1}{\pi} \left| \frac{1}{4} \sum_k \mathcal{T}_{kk} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) + \mathcal{F}_2 \left( \frac{\partial}{\partial \bar{z}_i^0}, Z^\perp \right) \right|^2 \right] G^\perp(Z^\perp)^2.$$

By Theorem 3.1, (3.18), (5.93),  $\mathcal{F}_j G^\perp$ , ( $j = 1, 3$ ),  $\mathcal{F}_2(\frac{\partial}{\partial \bar{z}_i^0}, \cdot) G^\perp$  are eigenfunctions of  $\mathcal{L}^\perp$  with eigenvalues  $4\pi j$ ,  $8\pi$ , thus they are orthogonal to each other.

From (5.76), (5.95)-(5.97), we have

$$(5.99) \quad \Psi_{1,4}(Z^\perp) = G^\perp(Z^\perp)^2 \int_{\mathbb{R}^{n_0}} \left\{ \left( (\mathcal{F}_1 G^\perp)(Z'^\perp) \right)^2 + \left( (\mathcal{F}_3 G^\perp)(Z'^\perp) \right)^2 \right. \\ \left. + \frac{1}{16\pi} \left| \sum_k \mathcal{T}_{kk} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) G^\perp \right|^2(Z'^\perp) + \frac{1}{\pi} \left| \mathcal{F}_2 \left( \frac{\partial}{\partial \bar{z}_i^0}, \cdot \right) G^\perp \right|^2(Z'^\perp) \right\} dv_{NG}(Z'^\perp).$$

From (3.18), (5.76), (5.98), (5.99) and the above discussion, we get (5.93).  $\square$

Now we need to compute the contribution from  $-(\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{O}_2 P^N$ .

**Lemma 5.11.** *The following identity holds,*

$$(5.100) \quad \tilde{\Psi}_{1,2}(Z^\perp) = \left\{ \frac{1}{2\pi} \left\langle R^{TX_G} \left( \frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_i^0} \right) \frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle + \frac{1}{48\pi} \left\langle R^{TB} \left( e_k^\perp, \frac{\partial}{\partial z_j^0} \right) e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle \right. \\ + \frac{1}{96\pi} \left| T \left( \frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0} \right) \right|^2 - \frac{\sqrt{-1}}{16\pi} \left\langle T(e_k^\perp, J e_k^\perp), T \left( \frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0} \right) \right\rangle + \frac{13}{192\pi} \left| T(e_k^\perp, \frac{\partial}{\partial z_j^0}) \right|^2 \\ + \frac{\sqrt{-1}}{96\pi} \left\langle 11 \nabla_{\frac{\partial}{\partial z_j^0}}^{TY} \left( T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) \right) + 4 \nabla_{\frac{\partial}{\partial \bar{z}_j^0}}^{TY} \left( T(e_k^\perp, \frac{\partial}{\partial z_j^0}) \right) + 7 \nabla_{e_k^\perp}^{TY} \left( T \left( \frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0} \right) \right), J e_k^\perp \right\rangle \\ \left. - \frac{2}{3\pi} \nabla_{\frac{\partial}{\partial z_j^0}} \nabla_{\frac{\partial}{\partial \bar{z}_j^0}} \log h + \frac{1}{2\pi} R^{EB} \left( \frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0} \right) \right\} P_{\mathcal{L}^\perp}(Z^\perp, Z^\perp).$$

*Proof.* By (3.9), (3.12), (3.54), (5.23) and (5.82),

$$(5.101) \quad I_1 P^N = \left\{ \frac{1}{2} b_i^\perp B \left( Z, \frac{\partial}{\partial \bar{z}_i^0} \right) + b_j B \left( Z, \frac{\partial}{\partial \bar{z}_j^0} \right) + \frac{\partial}{\partial z_j^0} \left( B \left( Z, \frac{\partial}{\partial \bar{z}_j^0} \right) \right) - \frac{\partial}{\partial \bar{z}_j^0} \left( B \left( Z, \frac{\partial}{\partial z_j^0} \right) \right) \right\} P^N.$$

By (3.55) and (5.101),

$$(5.102) \quad P_{\mathcal{L}^\perp} I_1 P^N = P_{\mathcal{L}^\perp} \left\{ b_j B \left( Z, \frac{\partial}{\partial \bar{z}_j^0} \right) + \frac{\partial}{\partial z_j^0} \left( B \left( Z, \frac{\partial}{\partial \bar{z}_j^0} \right) \right) - \frac{\partial}{\partial \bar{z}_j^0} \left( B \left( Z, \frac{\partial}{\partial z_j^0} \right) \right) \right\} P^N.$$

By (5.45b), and observe that from Theorem 3.1, only the monomials which have even degree on  $Z^\perp$  and  $\nabla_{e_j^\perp}$ , and which have also strictly positive degree on  $Z^0$  and  $\nabla_{0, e_j^0}$ , have contributions in  $P^{N^\perp} P_{\mathcal{L}^\perp} I_1 P^N$ .

By Remark 5.2, (3.55) and (5.45b),

$$(5.103) \quad P^{N^\perp} P_{\mathcal{L}^\perp} \left( \frac{\partial}{\partial z_j^0} \left( B \left( Z, \frac{\partial}{\partial \bar{z}_j^0} \right) \right) - \frac{\partial}{\partial \bar{z}_j^0} \left( B \left( Z, \frac{\partial}{\partial z_j^0} \right) \right) \right) P^N = -\pi \sqrt{-1} P^{N^\perp} P_{\mathcal{L}^\perp} \\ \frac{1}{6} \left\{ \frac{\partial}{\partial z_j^0} \left\langle R^{TX_G}(\mathcal{R}^0, J \mathcal{R}^0) \mathcal{R}^0, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle - \frac{\partial}{\partial \bar{z}_j^0} \left\langle R^{TX_G}(\mathcal{R}^0, J \mathcal{R}^0) \mathcal{R}^0, \frac{\partial}{\partial z_j^0} \right\rangle \right\} P^N \\ = -\frac{\pi}{3} P^{N^\perp} \left\langle 2 R^{TX_G}(z^0, \bar{z}^0) \frac{\partial}{\partial z_j^0} + R^{TX_G} \left( \frac{\partial}{\partial z_j^0}, \mathcal{R}^0 \right) z^0 + R^{TX_G} \left( \frac{\partial}{\partial z_j^0}, \bar{z}^0 \right) \mathcal{R}^0, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle P^N.$$



By (5.22), (5.92) and (5.103),

$$\begin{aligned}
 (5.104) \quad & - \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} P_{\mathcal{L}^\perp} \left( \frac{\partial}{\partial z_j^0} \left( B(Z, \frac{\partial}{\partial \bar{z}_j^0}) \right) - \frac{\partial}{\partial \bar{z}_j^0} \left( B(Z, \frac{\partial}{\partial z_j^0}) \right) \right) P^N \right) ((0, Z^\perp), (0, Z^\perp)) \\
 & = -\frac{1}{6\pi} \left\langle R^{TX_G} \left( \frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_i^0} \right) \frac{\partial}{\partial z_j^0} + R^{TX_G} \left( \frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_i^0} \right) \frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle P_{\mathcal{L}^\perp}(Z^\perp, Z^\perp).
 \end{aligned}$$

Observe that if  $Q$  is an odd degree monomial on  $b_j, b_j^+, z_j^0, \bar{z}_j^0$ , then

$$(5.105) \quad (QP^N)((0, Z^\perp), (0, Z'^\perp)) = 0.$$

By using this observation and (5.45b), we get

$$\begin{aligned}
 (5.106) \quad & - \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} b_j B(Z, \frac{\partial}{\partial \bar{z}_j^0}) P^N \right) ((0, Z^\perp), (0, Z'^\perp)) \\
 & = \pi \sqrt{-1} \left\{ (\mathcal{L}_2^0)^{-1} P^{N^\perp} b_j \left[ \frac{1}{6} \left\langle R^{TX_G}(\mathcal{R}^0, J\mathcal{R}^0) \mathcal{R}^0, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle \right. \right. \\
 & \quad - \frac{5}{4} \left\langle \nabla_{\mathcal{R}^0}^{TY} (T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})) Z_k^\perp + \nabla_{\mathcal{R}^\perp}^{TY} (T(e_k^0, \frac{\partial}{\partial \bar{z}_j^0})) Z_k^0, J\mathcal{R}^\perp \right\rangle \\
 & \quad + \left\langle \frac{1}{2} R^{TB}(\mathcal{R}^\perp, J\mathcal{R}^0) \mathcal{R}^\perp + \sqrt{-1} R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0) \mathcal{R}^\perp, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle \\
 & \quad \left. - \frac{3}{8} \sqrt{-1} \langle J\mathcal{R}^\perp, T(\mathcal{R}^0, e_i^0) \rangle \langle J\mathcal{R}^\perp, T(e_i^0, \frac{\partial}{\partial \bar{z}_j^0}) \rangle \right. \\
 & \quad \left. - \frac{1}{8} \left\langle T(\mathcal{R}^\perp, J\mathcal{R}^0), T(\mathcal{R}^\perp, \frac{\partial}{\partial \bar{z}_j^0}) \right\rangle + \frac{1}{2} \left\langle T(\mathcal{R}^\perp, J\mathcal{R}^\perp), T(\mathcal{R}^0, \frac{\partial}{\partial \bar{z}_j^0}) \right\rangle \right. \\
 & \quad \left. - \frac{1}{8} \left\langle JT(\frac{\partial}{\partial \bar{z}_j^0}, J\mathcal{R}^0), e_j^\perp \right\rangle \langle J\mathcal{R}^\perp, T(\mathcal{R}^\perp, e_j^\perp) \rangle \right] P^N \left\} ((0, Z^\perp), (0, Z'^\perp)).
 \end{aligned}$$

From (3.6), (3.54), (5.5b) and (5.83), we have

$$(5.107a) \quad \left\langle T\left(\frac{\partial}{\partial z_j^0}, e_i^0\right), T\left(e_i^0, \frac{\partial}{\partial \bar{z}_j^0}\right) \right\rangle = -2 \left| T\left(\frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0}\right) \right|^2,$$

$$(5.107b) \quad P_{\mathcal{L}^\perp} Z_k^\perp Z_l^\perp P_{\mathcal{L}^\perp} = \frac{\delta_{kl}}{4\pi} P_{\mathcal{L}^\perp}.$$

By (3.54), (5.5e), (5.92), (5.106) and (5.107a),

$$\begin{aligned}
(5.108) \quad & - \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} P_{\mathcal{L}^\perp} b_j B(Z, \frac{\partial}{\partial \bar{z}_j^0}) P^N \right) ((0, Z^\perp), (0, Z^\perp)) \\
& = \left\{ (\mathcal{L}_2^0)^{-1} P^{N^\perp} b_j \left[ \frac{\pi}{3} \left\langle R^{TX_G}(z^0, \bar{z}^0) \mathcal{R}^0, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle \right. \right. \\
& \quad - \frac{5\sqrt{-1}}{16} \left\langle \nabla_{\mathcal{R}^0}^{TY} (T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})) + \nabla_{e_k^\perp}^{TY} (T(e_i^0, \frac{\partial}{\partial \bar{z}_j^0})) Z_i^0, J e_k^\perp \right\rangle \\
& \quad + \frac{1}{8} \left\langle \sqrt{-1} R^{TB}(e_k^\perp, J \mathcal{R}^0) e_k^\perp - 2 R^{TB}(e_k^\perp, \mathcal{R}^0) e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle \\
& \quad + \frac{3}{32} \left\langle T(\mathcal{R}^0, e_i^0), T(e_i^0, \frac{\partial}{\partial \bar{z}_j^0}) \right\rangle - \frac{\sqrt{-1}}{32} \left\langle T(e_k^\perp, J \mathcal{R}^0), T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) \right\rangle \\
& \quad \left. + \frac{\sqrt{-1}}{8} \left\langle T(e_k^\perp, J e_k^\perp), T(\mathcal{R}^0, \frac{\partial}{\partial \bar{z}_j^0}) \right\rangle \right] P^N \left. \right\} ((0, Z^\perp), (0, Z^\perp)) \\
& = \left\{ - \frac{1}{12\pi} \left\langle R^{TX_G}(\frac{\partial}{\partial \bar{z}_j^0}, \frac{\partial}{\partial \bar{z}_i^0}) \frac{\partial}{\partial \bar{z}_i^0} + R^{TX_G}(\frac{\partial}{\partial \bar{z}_i^0}, \frac{\partial}{\partial \bar{z}_j^0}) \frac{\partial}{\partial \bar{z}_j^0}, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle \right. \\
& \quad + \frac{5\sqrt{-1}}{32\pi} \left\langle \nabla_{\frac{\partial}{\partial \bar{z}_j^0}}^{TY} (T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})) + \nabla_{e_k^\perp}^{TY} (T(\frac{\partial}{\partial \bar{z}_j^0}, \frac{\partial}{\partial \bar{z}_j^0})), J e_k^\perp \right\rangle \\
& \quad + \frac{3}{16\pi} \left\langle R^{TB}(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle + \frac{3}{32\pi} \left| T(\frac{\partial}{\partial \bar{z}_i^0}, \frac{\partial}{\partial \bar{z}_j^0}) \right|^2 \\
& \quad \left. - \frac{1}{64\pi} \left| T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) \right|^2 - \frac{\sqrt{-1}}{16\pi} \left\langle T(e_k^\perp, J e_k^\perp), T(\frac{\partial}{\partial \bar{z}_j^0}, \frac{\partial}{\partial \bar{z}_j^0}) \right\rangle \right\} P_{\mathcal{L}^\perp}(Z^\perp, Z^\perp).
\end{aligned}$$

For  $G_1(Z)$  (resp.  $G_2(Z)$ ) polynomials on  $Z$  with degree 1 (resp. 2) and  $F \in T_{x_0}^* X_G \otimes T_{x_0}^* X_G$ , by Theorem 3.1, (3.9), (3.12), (3.19), (3.54) and (3.55), for any  $k, l, k', l'$ ,

$$\begin{aligned}
(5.109) \quad & \nabla_{0, e_j^\perp} P^N = -2\pi Z_j^\perp P^N, \\
& P^{N^\perp} P_{\mathcal{L}^\perp} (G_1(Z) b_k^\perp + G_2(Z) b_k^\perp b_l^\perp + Z_{k'}^\perp b_{l'}) P^N = 0, \\
& \frac{1}{3} \left\langle R^{TB}(\mathcal{R}^\perp, e_i^\perp) \mathcal{R}^\perp, e_j^\perp \right\rangle \nabla_{0, e_i^\perp} \nabla_{0, e_j^\perp} P^N = -\frac{2\pi}{3} \left\langle R^{TB}(\mathcal{R}^\perp, e_j^\perp) \mathcal{R}^\perp, e_i^\perp \right\rangle P^N, \\
& F(e_i^0, e_j^0) \nabla_{0, e_i^0} \nabla_{0, e_j^0} P^N = \left[ F(\frac{\partial}{\partial \bar{z}_i^0}, \frac{\partial}{\partial \bar{z}_j^0}) b_i b_j - 4\pi F(\frac{\partial}{\partial \bar{z}_j^0}, \frac{\partial}{\partial \bar{z}_i^0}) \right] P^N.
\end{aligned}$$

By (5.23), (5.109), we get

$$\begin{aligned}
(5.110) \quad & I_2 P^N = \left\{ \left( \left\langle \frac{1}{3} R^{TX_G}(\mathcal{R}^0, \frac{\partial}{\partial \bar{z}_i^0}) \mathcal{R}^0 + R^{TB}(\mathcal{R}^\perp, \frac{\partial}{\partial \bar{z}_i^0}) \mathcal{R}^\perp + \nabla_{\mathcal{R}^0}^{TX_G} (A(\frac{\partial}{\partial \bar{z}_i^0}) \mathcal{R}^\perp), \frac{\partial}{\partial \bar{z}_j^0} \right\rangle \right. \right. \\
& \quad - 3 \left\langle A(\frac{\partial}{\partial \bar{z}_i^0}) \mathcal{R}^\perp, A(\frac{\partial}{\partial \bar{z}_j^0}) \mathcal{R}^\perp \right\rangle + \left\langle \frac{\partial}{\partial \bar{z}_i^0}, \nabla_{\mathcal{R}^0}^{TX_G} (A(\frac{\partial}{\partial \bar{z}_j^0}) \mathcal{R}^\perp) \right\rangle \left. \right) b_i b_j \\
& \quad - 4\pi \left\langle \frac{1}{3} R^{TX_G}(\mathcal{R}^0, \frac{\partial}{\partial \bar{z}_j^0}) \mathcal{R}^0 + R^{TB}(\mathcal{R}^\perp, \frac{\partial}{\partial \bar{z}_j^0}) \mathcal{R}^\perp + \nabla_{\mathcal{R}^0}^{TX_G} (A(\frac{\partial}{\partial \bar{z}_j^0}) \mathcal{R}^\perp), \frac{\partial}{\partial \bar{z}_i^0} \right\rangle \\
& \quad \left. + 12\pi |A(\frac{\partial}{\partial \bar{z}_j^0}) \mathcal{R}^\perp|^2 - 4\pi \left\langle \frac{\partial}{\partial \bar{z}_j^0}, \nabla_{\mathcal{R}^0}^{TX_G} (A(\frac{\partial}{\partial \bar{z}_i^0}) \mathcal{R}^\perp) \right\rangle - \frac{2\pi}{3} \left\langle R^{TB}(\mathcal{R}^\perp, e_j^\perp) \mathcal{R}^\perp, e_i^\perp \right\rangle \right\} P^N.
\end{aligned}$$

Observe that as  $A(e_i^0) e_i^0 \in N_G$ , we have at  $x_0$ ,

$$(5.111) \quad \left\langle \nabla_{\mathcal{R}^0}^{TB} (A(e_i^0) e_i^0), e_j^0 \right\rangle = \left\langle A(\mathcal{R}^0) A(e_i^0) e_i^0, e_j^0 \right\rangle.$$

Thus by (3.12), (3.54), (3.55), (5.24), (5.107b), (5.109)-(5.111),  $a_j = a_j^\perp = 2\pi$ , and the arguments above (5.103),

$$(5.112a) \quad P^{N^\perp} P_{\mathcal{L}^\perp} \langle \Gamma_{ii}(\mathcal{R}), e_l \rangle \nabla_{0, e_l} P^N = -\frac{2}{3} P^{N^\perp} \left\langle R^{TXG}(\mathcal{R}^0, e_i^0) e_i^0, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle b_j P^N,$$

$$(5.112b) \quad P^{N^\perp} P_{\mathcal{L}^\perp} I_2 P^N = P^{N^\perp} \left\{ \left( \left\langle \frac{1}{3} R^{TXG}(\mathcal{R}^0, \frac{\partial}{\partial \bar{z}_i^0}) \mathcal{R}^0 + \frac{1}{4\pi} R^{TB}(e_k^\perp, \frac{\partial}{\partial \bar{z}_i^0}) e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle \right. \right. \\ \left. \left. - \frac{3}{4\pi} \left\langle A(\frac{\partial}{\partial \bar{z}_i^0}) e_k^\perp, A(\frac{\partial}{\partial \bar{z}_j^0}) e_k^\perp \right\rangle \right) b_i b_j - \frac{4\pi}{3} \left\langle R^{TXG}(\mathcal{R}^0, \frac{\partial}{\partial \bar{z}_j^0}) \mathcal{R}^0, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle \right\} P^N.$$

By (3.6), (5.4), (5.92), (5.112a), (5.112b) and the fact that  $R^{TXG}(\cdot, \cdot)$  is  $(1, 1)$ -form, we get

$$(5.113) \quad - \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} P_{\mathcal{L}^\perp} (I_2 + \langle \Gamma_{ii}(\mathcal{R}), e_l \rangle \nabla_{0, e_l}) P^N \right) ((0, Z^\perp), (0, Z^\perp)) \\ = \frac{1}{6\pi} \left\{ 3 \left\langle R^{TXG}(\frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_i^0}) \frac{\partial}{\partial z_j^0} + R^{TXG}(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_i^0}) \frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle \right. \\ \left. - 2 \left\langle R^{TXG}(\frac{\partial}{\partial z_j^0}, e_i^0) e_i^0 + R^{TXG}(\frac{\partial}{\partial \bar{z}_i^0}, \frac{\partial}{\partial \bar{z}_j^0}) \frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle \right\} P_{\mathcal{L}^\perp}(Z^\perp, Z^\perp) \\ = \frac{2}{3\pi} \left\langle R^{TXG}(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_i^0}) \frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle P_{\mathcal{L}^\perp}(Z^\perp, Z^\perp).$$

Now by (5.45a), (5.83), (5.107b) and (5.109),

$$(5.114) \quad - P^{N^\perp} P_{\mathcal{L}^\perp} \frac{1}{9} \sum_i \left[ \sum_j (\partial_j R^{LB})_{x_0}(\mathcal{R}, e_i) Z_j \right]^2 P^N = \frac{\pi}{4} P^{N^\perp} \left| T(\mathcal{R}^0, e_i^\perp) \right|^2 P^N, \\ - \frac{1}{4} (\mathcal{L}_2^0)^{-1} P^{N^\perp} P_{\mathcal{L}^\perp} [K_2(\mathcal{R}), \mathcal{L}_2^0] P^N = \frac{1}{4} P^{N^\perp} P_{\mathcal{L}^\perp} K_2(\mathcal{R}) P^N \\ = \frac{1}{12} P^{N^\perp} \left\langle R^{TXG}(\mathcal{R}^0, e_i^0) \mathcal{R}^0, e_i^0 \right\rangle P^N.$$

By (5.13), (5.46), (5.48) and (5.49), we get

$$(5.115) \quad \frac{\sqrt{-1}}{2\pi} (\partial_j^0 R^{LB})_{x_0}(\mathcal{R}, e_i^0) = -\frac{1}{2} \langle J\mathcal{R}^\perp, T(e_j^0, e_i^0) \rangle + \langle JA(e_j^0) \mathcal{R}^\perp, e_i^0 \rangle = 0.$$

Thus by (3.9), (5.26), (5.45a), (5.114) and (5.115), we get

$$(5.116) \quad - P^{N^\perp} P_{\mathcal{L}^\perp} \mathcal{O}'_2 P^N = P^{N^\perp} P_{\mathcal{L}^\perp} \left\{ -I_1 - (I_2 + \langle \Gamma_{ii}(\mathcal{R}), e_l \rangle \nabla_{0, e_l}) \right. \\ \left. - \frac{1}{4} [K_2(\mathcal{R}), \mathcal{L}_2^0] - R^{EB}(\mathcal{R}, \frac{\partial}{\partial \bar{z}_j^0}) b_j - \frac{\pi}{4} \left| T(\mathcal{R}^0, e_i^\perp) \right|^2 \right\} P^N.$$

Note that  $R^{TXG}(\cdot, \cdot)$  is a  $(1, 1)$ -form, by (3.54), (5.4), (5.92), (5.102), (5.104), (5.108), (5.113) and (5.116),

$$\begin{aligned}
 (5.117) \quad & - \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} P_{\mathcal{L}^\perp} \mathcal{O}_2' P^N \right) ((0, Z^\perp), (0, Z^\perp)) \\
 & = - \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} P_{\mathcal{L}^\perp} (I_1 + I_2 + \langle \Gamma_{ii}(\mathcal{R}), e_i \rangle \nabla_{0, e_i}) P^N \right) ((0, Z^\perp), (0, Z^\perp)) \\
 & + \frac{1}{2\pi} \left\{ R^{EB} \left( \frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0} \right) + \frac{1}{3} \left\langle R^{TXG} \left( \frac{\partial}{\partial z_j^0}, e_i^0 \right) e_i^0, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle + \frac{1}{4} \left| T \left( \frac{\partial}{\partial z_j^0}, e_i^\perp \right) \right|^2 \right\} P_{\mathcal{L}^\perp} (Z^\perp, Z^\perp) \\
 & = \left\{ \frac{1}{2\pi} \left\langle R^{TXG} \left( \frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_i^0} \right) \frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle + \frac{3}{16\pi} \left\langle R^{TB} (e_k^\perp, \frac{\partial}{\partial z_j^0}) e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle \right. \\
 & + \frac{3}{32\pi} \left| T \left( \frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0} \right) \right|^2 - \frac{\sqrt{-1}}{16\pi} \left\langle T(e_k^\perp, J e_k^\perp), T \left( \frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0} \right) \right\rangle + \frac{7}{64\pi} \left| T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) \right|^2 \\
 & \left. + \frac{5\sqrt{-1}}{32\pi} \left\langle \nabla_{\frac{\partial}{\partial z_j^0}}^{TY} (T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})) + \nabla_{e_k^\perp}^{TY} (T(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0})), J e_k^\perp \right\rangle + \frac{1}{2\pi} R^{EB} \left( \frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0} \right) \right\} P_{\mathcal{L}^\perp} (Z^\perp, Z^\perp).
 \end{aligned}$$

By (3.54), (5.62), (5.83), (5.107b), (5.109) and the arguments above (5.103),

$$\begin{aligned}
 (5.118) \quad & 4\pi^2 P^{N^\perp} P_{\mathcal{L}^\perp} \mathcal{O}_2'' P^N = 4\pi^2 P^{N^\perp} P_{\mathcal{L}^\perp} \left\{ -\frac{1}{3} \langle (\nabla_{\cdot}^{TY} \dot{g}^{TY})_{(\mathcal{R}^0, \mathcal{R}^0)} J \mathcal{R}^\perp, J \mathcal{R}^\perp \rangle \right. \\
 & + \frac{1}{6} \langle \nabla_{\mathcal{R}^0}^{TY} (T(e_j^\perp, J_{x_0} e_i^0)) Z_j^\perp Z_i^0 + \nabla_{\mathcal{R}^\perp}^{TY} (T(e_j^0, J_{x_0} e_i^0)) Z_j^0 Z_i^0, J \mathcal{R}^\perp \rangle \\
 & + \frac{1}{3} \langle R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0) \mathcal{R}^0, \mathcal{R}^\perp \rangle - \frac{1}{12} \sum_l \langle T(\mathcal{R}^0, e_l), J \mathcal{R}^\perp \rangle^2 \left. \right\} P^N \\
 & = \frac{\pi}{3} P^{N^\perp} \left\{ \frac{1}{2} \left\langle \nabla_{\mathcal{R}^0}^{TY} (T(e_k^\perp, J_{x_0} e_i^0)) Z_i^0 + \nabla_{e_k^\perp}^{TY} (T(e_j^0, J_{x_0} e_i^0)) Z_j^0 Z_i^0, J e_k^\perp \right\rangle \right. \\
 & \left. - \langle (\nabla_{\cdot}^{TY} \dot{g}^{TY})_{(\mathcal{R}^0, \mathcal{R}^0)} J e_k^\perp, J e_k^\perp \rangle + \langle R^{TB}(e_k^\perp, \mathcal{R}^0) \mathcal{R}^0, e_k^\perp \rangle - \frac{1}{4} |T(\mathcal{R}^0, e_l)|^2 \right\} P^N.
 \end{aligned}$$

Let  $\{f_l\}$  be an orthonormal frame of  $TY$  on  $X$ .

As  $\nabla^{TY}$  preserves the metric  $g^{TY}$ , by (1.4), (1.24),

$$(5.119) \quad \left\langle (\nabla_{e_i^0}^{TY} \dot{g}_{e_j^0}^{TY}) f_l, f_l \right\rangle = \nabla_{e_i^0} \left\langle \dot{g}_{e_j^0}^{TY} f_l, f_l \right\rangle = 4 \nabla_{e_i^0} \nabla_{e_j^0} \log h.$$

Now  $\{J e_k^\perp\}$  is an orthonormal basis of  $TY$  along the fiber  $Y_{x_0}$  and  $\{e_l\} = \{e_i^0\} \cup \{e_k^\perp\}$ .

By (3.54), (5.92), (5.107a), (5.118) and (5.119),

$$\begin{aligned}
 (5.120) \quad & - 4\pi^2 \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} P_{\mathcal{L}^\perp} \mathcal{O}_2'' P^N \right) ((0, Z^\perp), (0, Z^\perp)) \\
 & = \frac{1}{4\pi} \left\{ \frac{\sqrt{-1}}{6} \left\langle -\nabla_{\frac{\partial}{\partial z_j^0}}^{TY} (T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})) + \nabla_{\frac{\partial}{\partial \bar{z}_j^0}}^{TY} (T(e_k^\perp, \frac{\partial}{\partial z_j^0})) - 2 \nabla_{e_k^\perp}^{TY} (T(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0})), J e_k^\perp \right\rangle \right. \\
 & - \frac{8}{3} \nabla_{\frac{\partial}{\partial z_j^0}} \nabla_{\frac{\partial}{\partial \bar{z}_j^0}} \log h - \frac{1}{3} \left| T(\frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0}) \right|^2 - \frac{1}{6} \left| T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) \right|^2 \\
 & \left. - \frac{2}{3} \left\langle R^{TB}(e_k^\perp, \frac{\partial}{\partial z_j^0}) e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle \right\} P_{\mathcal{L}^\perp} (Z^\perp, Z^\perp).
 \end{aligned}$$

By (5.73), (5.76), (5.117) and (5.120), we get (5.100). The proof of Lemma 5.11 is complete.  $\square$

**5.4. Final computations: the proof of Theorem 0.6.** By (3.40), (5.3), (5.5a), (5.6a) and (5.31), as  $Je_k^\perp \in TY$  on  $P$ , we get at  $x_0$ ,

$$(5.121) \quad \begin{aligned} \nabla_{e_i^0}^{TY} Je_k^\perp &= P^{TY} \nabla_{e_i^0}^{TX} Je_k^\perp = P^{TY} J \nabla_{e_i^0}^{TX} e_k^\perp = 0, \\ \nabla_{e_i^0}^{TB} Je_j^0 &= \nabla_{e_i^0}^{TX_G} Je_j^0 + A(e_i^0) Je_j^0 = -\frac{1}{2} JT(e_i^0, e_j^0) = \nabla_{e_i^0}^{TB} (J_{x_0} e_j^0). \end{aligned}$$

By (1.6), (1.24), (5.5c) and (5.121), as in (5.119), at  $x_0$ ,

$$(5.122) \quad \begin{aligned} \left\langle \nabla_{e_i^0}^{TY} (T(e_k^\perp, e_j^0)), Je_k^\perp \right\rangle_{x_0} &= -2 \left\langle \nabla_{e_i^0}^{TY} (T(Je_j^0, Je_k^\perp)), Je_k^\perp \right\rangle \\ &= - \left\langle (\nabla_{e_i^0}^{TY} \dot{g}_{Je_j^0}^{TY}) Je_k^\perp, Je_k^\perp \right\rangle = -4 \nabla_{e_i^0} \nabla_{J_{x_0} e_j^0} \log h. \end{aligned}$$

By (1.21) and (5.122), we get

$$(5.123) \quad \begin{aligned} \sqrt{-1} \left\langle \nabla_{\frac{\partial}{\partial z_j^0}}^{TY} (T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})), Je_k^\perp \right\rangle &= -4 \nabla_{\frac{\partial}{\partial z_j^0}} \nabla_{\frac{\partial}{\partial \bar{z}_j^0}} \log h = \Delta_{X_G} \log h, \\ \sqrt{-1} \left\langle \nabla_{\frac{\partial}{\partial \bar{z}_j^0}}^{TY} (T(e_k^\perp, \frac{\partial}{\partial z_j^0})), Je_k^\perp \right\rangle &= -\Delta_{X_G} \log h. \end{aligned}$$

Note that  $T(e_i, e_j) = -[e_i^H, e_j^H]$ , as  $[e_i, e_j] = 0$ . By (1.4) and (1.6),

$$(5.124) \quad \begin{aligned} \nabla_{e_k^{\perp, H}}^{TY} (T(e_i^{0, H}, e_j^{0, H})) &= -[e_k^{\perp, H}, [e_i^{0, H}, e_j^{0, H}]] + T(e_k^{\perp, H}, T(e_i^{0, H}, e_j^{0, H})) \\ &= L_{e_i^{0, H}} (T(e_k^{\perp, H}, e_j^{0, H})) - L_{e_j^{0, H}} (T(e_k^{\perp, H}, e_i^{0, H})) + T(e_k^{\perp, H}, T(e_i^{0, H}, e_j^{0, H})) \\ &= \nabla_{e_i^{0, H}}^{TY} (T(e_k^{\perp, H}, e_j^{0, H})) - \nabla_{e_j^{0, H}}^{TY} (T(e_k^{\perp, H}, e_i^{0, H})) - T(e_i^{0, H}, T(e_k^{\perp, H}, e_j^{0, H})) \\ &\quad + T(e_j^{0, H}, T(e_k^{\perp, H}, e_i^{0, H})) + T(e_k^{\perp, H}, T(e_i^{0, H}, e_j^{0, H})). \end{aligned}$$

Thus by Theorem 5.1, (5.123) and (5.124),

$$(5.125) \quad \begin{aligned} \sqrt{-1} \left\langle \nabla_{e_k^\perp}^{TY} (T(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0})), Je_k^\perp \right\rangle &= \sqrt{-1} \left\{ 2 \left\langle \nabla_{\frac{\partial}{\partial z_j^0}}^{TY} (T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})), Je_k^\perp \right\rangle \right. \\ &\quad \left. - \left\langle T(\frac{\partial}{\partial z_j^0}, Je_k^\perp), T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) \right\rangle + \left\langle T(\frac{\partial}{\partial \bar{z}_j^0}, Je_k^\perp), T(e_k^\perp, \frac{\partial}{\partial z_j^0}) \right\rangle + \left\langle T(e_k^\perp, Je_k^\perp), T(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0}) \right\rangle \right\} \\ &= 2 \Delta_{X_G} \log h + |T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})|^2 + \sqrt{-1} \left\langle T(e_k^\perp, Je_k^\perp), T(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0}) \right\rangle. \end{aligned}$$

By  $T(e_i, e_j) = -[e_i^H, e_j^H]$ , (3.40) and (5.54), we have

$$(5.126) \quad \begin{aligned} R^{TX} (e_k^H, e_j^H) e_i^H &= \nabla_{e_k^H}^{TX} \nabla_{e_j^H}^{TX} e_i^H - \nabla_{e_j^H}^{TX} \nabla_{e_k^H}^{TX} e_i^H - \nabla_{[e_k^H, e_j^H]}^{TX} e_i^H \\ &= R^{TB} (e_k, e_j) e_i - \frac{1}{2} T(e_k, \nabla_{e_j}^{TB} e_i) + \frac{1}{2} T(e_j, \nabla_{e_k}^{TB} e_i) \\ &\quad - \frac{1}{2} \nabla_{e_k^H}^{TX} (T(e_j, e_i)) + \frac{1}{2} \nabla_{e_j^H}^{TX} (T(e_k, e_i)) + \nabla_{T(e_k^H, e_j^H)}^{TX} e_i^H, \\ \left\langle R^{TX} (e_k^{\perp, H}, e_j^{0, H}) (J_{x_0} e_j^0)^H, J_{x_0} e_k^{\perp, H} \right\rangle &= \left\langle R^{TX} (e_k^{\perp, H}, e_j^{0, H}) e_j^{0, H}, e_k^{\perp, H} \right\rangle. \end{aligned}$$

By (5.5a), (5.6a), (5.13), (5.121) and  $T(e_k^\perp, e_i^0) \in TY$ , at  $x_0$ , we get

$$(5.127) \quad \begin{aligned} \nabla_{e_j^0}^{TB}(J_{x_0}e_j^0) &= 0, \quad \nabla_{e_k^\perp}^{TB}(J_{x_0}e_j^0) = \frac{1}{2} \langle T(e_j^0, e_l^0), J e_k^\perp \rangle e_l^0, \\ \left\langle \nabla_{T(e_k^\perp, e_i^0)}^{TX}(J_{x_0}e_i^0)^H, J_{x_0}e_k^\perp \right\rangle &= \left\langle \nabla_{T(e_k^\perp, e_i^0)}^{TX}e_i^{0,H}, e_k^\perp \right\rangle. \end{aligned}$$

We apply now the first equation of (5.126) into the second equation of (5.126), by using (1.8) and (5.133) and  $T(, )$  is  $(1, 1)$ -form, we get at  $x_0$ ,

$$(5.128) \quad \begin{aligned} \frac{1}{4}|T(e_j^0, e_l^0)|^2 - \frac{1}{2} \left\langle \nabla_{e_k^\perp}^{TY}(T(e_j^0, J_{x_0}e_j^0)), J e_k^\perp \right\rangle + \frac{1}{2} \left\langle \nabla_{e_j^0}^{TY}(T(e_k^\perp, J_{x_0}e_j^0)), J e_k^\perp \right\rangle \\ = \left\langle R^{TB}(e_k^\perp, e_j^0)e_j^0, e_k^\perp \right\rangle + \frac{1}{2} \left\langle \nabla_{e_j^0}^{TX}(T(e_k^\perp, e_j^0)), e_k^\perp \right\rangle \\ = \left\langle R^{TB}(e_k^\perp, e_j^0)e_j^0, e_k^\perp \right\rangle - \frac{1}{4}|T(e_k^\perp, e_l^0)|^2. \end{aligned}$$

Finally from (3.6), (5.123), (5.125) and (5.128) and  $T(, )$  is  $(1, 1)$ -form, we get

$$(5.129) \quad \begin{aligned} 2 \left\langle R^{TB}(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) \frac{\partial}{\partial \bar{z}_j^0}, e_k^\perp \right\rangle &= \sqrt{-1} \left\langle \nabla_{e_k^\perp}^{TY}(T(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0})), J e_k^\perp \right\rangle \\ &\quad - \sqrt{-1} \left\langle \nabla_{\frac{\partial}{\partial \bar{z}_j^0}}^{TY}(T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})), J e_k^\perp \right\rangle + \frac{1}{2}|T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})|^2 + \left| T(\frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0}) \right|^2 \\ &= \Delta_{X_G} \log h + \frac{3}{2}|T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})|^2 + \left| T(\frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0}) \right|^2 + \sqrt{-1} \left\langle T(e_k^\perp, J e_k^\perp), T(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0}) \right\rangle. \end{aligned}$$

From (5.123)-(5.129),

$$(5.130) \quad \begin{aligned} \frac{\sqrt{-1}}{96\pi} \left\langle 11 \nabla_{\frac{\partial}{\partial z_j^0}}^{TY}(T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})) + 4 \nabla_{\frac{\partial}{\partial \bar{z}_j^0}}^{TY}(T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})) + 7 \nabla_{e_k^\perp}^{TY}(T(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0})), J e_k^\perp \right\rangle \\ + \frac{1}{48\pi} \left\langle R^{TB}(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle - \frac{\sqrt{-1}}{16\pi} \left\langle T(e_k^\perp, J e_k^\perp), T(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0}) \right\rangle \\ = \frac{5}{24\pi} \Delta_{X_G} \log h + \frac{11}{192\pi} \left| T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) \right|^2 - \frac{1}{96\pi} \left| T(\frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0}) \right|^2. \end{aligned}$$

By (3.19), (5.76), (5.81), (5.100) and (5.130),

$$(5.131) \quad \begin{aligned} \Phi_{1,1} + \Phi_{1,2} &= \frac{1}{2\pi} \left\langle R^{TX_G}(\frac{\partial}{\partial \bar{z}_j^0}, \frac{\partial}{\partial \bar{z}_i^0}) \frac{\partial}{\partial \bar{z}_i^0}, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle + \frac{3}{8\pi} \Delta_{X_G} \log h + \frac{1}{2\pi} R^{E_B}(\frac{\partial}{\partial \bar{z}_j^0}, \frac{\partial}{\partial \bar{z}_j^0}) \\ &= \frac{1}{16\pi} r_{x_0}^{X_G} + \frac{3}{8\pi} \Delta_{X_G} \log h + \frac{1}{4\pi} R_{x_0}^{E_G}(w_j^0, \bar{w}_j^0). \end{aligned}$$

From Lemma 5.10, (5.80) and (5.131), we get (0.25).

Recall that we compute everything on  $\mathcal{C}^\infty(X, L^p \otimes E)$ .

From (5.18), (5.21), (5.22), comparing to (2.110), we know that in (0.20),  $\Phi_r(x_0) \in \text{End}(E_G)_{x_0}$ , and the term  $r^X$ ,  $R^{\det}$  will not appear here, and  $\tau = 2\pi n$ , thus we get the remainder part of Theorem 0.6 from Corollary 0.4.

The proof of Theorem 0.6 is complete.

5.5. **Coefficient**  $P^{(2)}(0, 0)$ . As in (5.80), we have

$$(5.132) \quad P^{(2)}(0, 0) = (\Psi_{1,1} + \Psi_{1,2})(0) + (\Psi_{1,1} + \Psi_{1,2})^*(0) + (\Psi_{1,3} - \Psi_{1,4})(0).$$

For  $k \in \mathbb{N}$ , let  $H_k(x)$  be the Hermite polynomial,

$$(5.133) \quad H_k(x) = \sum_{j=0}^{[k/2]} (-1)^j \frac{k! (2x)^{k-2j}}{j! (k-2j)!}.$$

By [38, §8.6], (3.8) and  $a_l^\perp = 2\pi$ , we have

$$(5.134) \quad (b_l^\perp)^k e^{-\pi|Z_l^\perp|^2} = (2\pi)^{k/2} H_k(\sqrt{2\pi} Z_l^\perp) e^{-\pi|Z_l^\perp|^2}.$$

Especially, for  $l$  fixed,  $i \in \mathbb{N}$ ,

$$(5.135) \quad \begin{aligned} ((b_l^\perp)^{2i+1} e^{-\pi|Z_l^\perp|^2})(0) &= 0, \\ ((b_l^\perp)^2 e^{-\pi|Z_l^\perp|^2})(0) &= -4\pi, \quad ((b_l^\perp)^4 e^{-\pi|Z_l^\perp|^2})(0) = 3 \cdot (4\pi)^2, \\ ((b_l^\perp)^6 e^{-\pi|Z_l^\perp|^2})(0) &= 15 \cdot (-4\pi)^3. \end{aligned}$$

Recall that when we meet the operation  $|\cdot|^2$ , we will first do this operation, then take the sum of the indices. Thus  $|\mathcal{T}_{ijk}|^2$  means  $\sum_{ijk} |\mathcal{T}_{ijk}|^2$ , etc.

By (3.22), (5.94) and (5.135),

$$(5.136) \quad \mathcal{F}_2(\cdot, 0) = -\frac{1}{8} \mathcal{T}_{kk}; \quad P^N(0, 0) = 2^{n_0/2}.$$

By (5.98), (5.135) and (5.136), we know

$$(5.137) \quad \Psi_{1,3}(0) = \frac{2^{n_0/2}}{\pi} \left| \frac{1}{4} \sum_k \mathcal{T}_{kk} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) + \mathcal{F}_2 \left( \frac{\partial}{\partial \bar{z}_i^0}, 0 \right) \right|^2 = \frac{2^{n_0/2}}{64\pi} \left| \sum_k \mathcal{T}_{kk} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) \right|^2.$$

and from (3.17), (3.18), (3.54), (5.99), (5.136) and  $a_i^\perp = 2\pi$ ,

$$(5.138) \quad \begin{aligned} \Psi_{1,4}(0) &= G^\perp(0)^2 \left\{ \frac{1}{4\pi} \sum_k \mathcal{F}_1(e_k^\perp)^2 + \frac{6 \cdot (4\pi)^3}{(192\pi^2)^2} |\mathcal{T}_{klm}|^2 \right. \\ &\quad \left. + \frac{1}{16\pi} \left| \sum_k \mathcal{T}_{kk} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) \right|^2 + \frac{2 \cdot (4\pi)^2}{\pi \cdot (32\pi)^2} \left| \mathcal{T}_{kl} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) \right|^2 \right\} \\ &= \frac{2^{n_0/2}}{4\pi} \left\{ \sum_k \mathcal{F}_1(e_k^\perp)^2 + \frac{1}{24} |\mathcal{T}_{klm}|^2 + \frac{1}{4} \left| \sum_k \mathcal{T}_{kk} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) \right|^2 + \frac{1}{8} \left| \mathcal{T}_{kl} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) \right|^2 \right\}. \end{aligned}$$

**Lemma 5.12.** *The following identity holds,*

$$(5.139) \quad \begin{aligned} \Psi_{1,1}(0) &= \left\{ -\frac{19}{2^6 \cdot 3\pi} |\mathcal{T}_{jj'} \left( \frac{\partial}{\partial \bar{z}_i^0} \right)|^2 - \frac{11}{2^7 \cdot 3\pi} \mathcal{T}_{klm}^2 + \frac{1}{2^8 \pi} \mathcal{T}_{kkm} \mathcal{T}_{llm} \right. \\ &\quad \left. - \frac{5}{2^7 \pi} \mathcal{T}_{jj} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) \mathcal{T}_{kk} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) - \frac{1}{8\pi} \sum_k \mathcal{F}_1(e_k^\perp)^2 - \frac{1}{16\pi} \mathcal{F}_1(e_k^\perp) \mathcal{T}_{kl} \right\} P^N(0, 0). \end{aligned}$$

*Proof.* Set

$$\begin{aligned}
 \mathcal{I}_1 &= -\sqrt{-1} \left( \mathcal{T}_{jj'} \left( \frac{\partial}{\partial z_i^0} \right) \frac{b_i^+}{8\pi} + \frac{1}{4} \mathcal{T}_{jj'}(z^0) \right) b_j^\perp b_{j'}^\perp \frac{\sqrt{-1}}{8\pi} b_l \mathcal{T}_{kk} \left( \frac{\partial}{\partial \bar{z}_l^0} \right) \\
 (5.140) \quad \mathcal{I}_2 &= \sqrt{-1} \left( \mathcal{T}_{jj'} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) b_i \frac{B_{jj'}^\perp}{8\pi} + \frac{1}{4} \mathcal{T}_{jj'}(\bar{z}^0) (b_j^{\perp+} b_{j'}^{\perp+} - b_j^\perp b_{j'}^\perp) \right) \frac{-\sqrt{-1}}{32\pi} \mathcal{T}_{kl}(z^0) b_k^\perp b_l^\perp, \\
 \mathcal{I}_3 &= -\frac{\sqrt{-1}}{8\pi} \tilde{\mathcal{T}}_{ijj'} (b_j^\perp b_{j'}^{\perp+} + b_j^\perp b_{j'}^\perp) b_i^{\perp+} \left( \mathcal{F}_1(e_k^\perp) \frac{b_k^\perp}{4\pi} + \mathcal{T}_{klm} \frac{b_k^\perp b_l^\perp b_m^\perp}{192\pi^2} \right).
 \end{aligned}$$

Observe that by (5.92), when we evaluate  $\Psi_{1,1}$  in (5.76), in each monomial, if the total degree of  $b_l$ ,  $\bar{z}^0$  is not as same as the total degree of  $b_l^\perp$ ,  $z^0$ , then the contribution of this term is 0. Thus by (3.9), (3.54), (5.76), (5.83), (5.86), (5.87), (5.94) and (5.140),

$$\begin{aligned}
 (5.141) \quad \Psi_{1,1}(Z^\perp) &= \left\{ (\mathcal{L}_2^0)^{-1} P^{N^\perp} \left[ \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 \right. \right. \\
 &\quad \left. \left. + \left( \mathcal{F}_1(e_j^\perp) (b_j^{\perp+} + b_j^\perp) + \mathcal{T}_{ijj'} \frac{B_{ijj'}^\perp}{16\pi} \right) \left( \mathcal{F}_1(e_k^\perp) \frac{b_k^\perp}{4\pi} + \mathcal{T}_{klm} \frac{b_k^\perp b_l^\perp b_m^\perp}{192\pi^2} \right) \right] P^N \right\} (Z^\perp, Z^\perp).
 \end{aligned}$$

By (3.8), (3.19) and (5.135),

$$(5.142) \quad (b_j z_i^0 P^N)(0, 0) = -2\delta_{ij} P^N(0, 0), \quad (b_k^\perp b_l^\perp b_j z_i^0 P^N)(0, 0) = 8\pi \delta_{ij} \delta_{kl} P^N(0, 0).$$

From Theorem 3.1, (3.9), (3.54), (5.135), (5.140) and (5.142),

$$\begin{aligned}
 (5.143) \quad &((\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{I}_1 P^N)(0, 0) \\
 &= \frac{1}{32\pi} \left\{ (\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{T}_{kk} \left( \frac{\partial}{\partial \bar{z}_l^0} \right) \left( 4\mathcal{T}_{jj'} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) b_j^\perp b_{j'}^\perp + b_l b_j^\perp b_{j'}^\perp \mathcal{T}_{jj'}(z^0) \right) P^N \right\} (0, 0) \\
 &= \frac{1}{32\pi} \mathcal{T}_{kk} \left( \frac{\partial}{\partial \bar{z}_l^0} \right) \left\{ \left( \mathcal{T}_{jj'} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) \frac{b_j^\perp b_{j'}^\perp}{2\pi} + \frac{b_l b_j^\perp b_{j'}^\perp}{12\pi} \mathcal{T}_{jj'}(z^0) \right) P^N \right\} (0, 0) \\
 &= -\frac{1}{24\pi} \mathcal{T}_{jj} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) \mathcal{T}_{kk} \left( \frac{\partial}{\partial \bar{z}_l^0} \right) P^N(0, 0).
 \end{aligned}$$

By (3.9), (3.54), (5.83) and (5.140),

$$\begin{aligned}
 (5.144) \quad &(P^{N^\perp} \mathcal{I}_2 P^N)(Z, (0, Z'^\perp)) = \frac{1}{28\pi^2} \left\{ P^{N^\perp} \mathcal{T}_{jj'} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) \right. \\
 &\quad \left[ b_i \mathcal{T}_{kl}(z^0) B_{jj'}^\perp + (b_j^{\perp+} b_{j'}^{\perp+} - b_j^\perp b_{j'}^\perp) \mathcal{T}_{kl}(z^0) b_i \right] b_k^\perp b_l^\perp P^N \left. \right\} (Z, (0, Z'^\perp)) \\
 &= \frac{1}{28\pi^2} \mathcal{T}_{jj'} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) \left\{ P^{N^\perp} \left[ b_i \mathcal{T}_{kl}(z^0) (2b_j^{\perp+} b_{j'}^{\perp+} + 2b_j^\perp b_{j'}^{\perp+} + 4\pi \delta_{jj'}) \right. \right. \\
 &\quad \left. \left. + 2\mathcal{T}_{kl} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) (b_j^{\perp+} b_{j'}^{\perp+} - b_j^\perp b_{j'}^\perp) \right] b_k^\perp b_l^\perp P^N \right\} (Z, (0, Z'^\perp)) \\
 &= \frac{1}{28\pi^2} \mathcal{T}_{jj'} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) \left\{ b_i \left( 64\pi^2 \mathcal{T}_{jj'}(z^0) + 16\pi \mathcal{T}_{kj'}(z^0) b_j^\perp b_k^\perp + 4\pi \delta_{jj'} \mathcal{T}_{kl}(z^0) b_k^\perp b_l^\perp \right) \right. \\
 &\quad \left. - 2\mathcal{T}_{kl} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) b_j^\perp b_{j'}^\perp b_k^\perp b_l^\perp P^N \right\} (Z, (0, Z'^\perp)).
 \end{aligned}$$



Thus by Theorem 3.1, (3.8), (5.135), (5.142), (5.144) and use a similar equation as (5.152) for  $\mathcal{T}_{jj'}(\frac{\partial}{\partial \bar{z}_i^0})\mathcal{T}_{kl}(\frac{\partial}{\partial z_i^0})b_j^\perp b_{j'}^\perp b_k^\perp b_l^\perp$ , we get

$$\begin{aligned}
 (5.145) \quad & ((\mathcal{L}_2^0)^{-1}P^{N^\perp}\mathcal{I}_2P^N)(0,0) = \frac{1}{28\pi^2}\mathcal{T}_{jj'}(\frac{\partial}{\partial \bar{z}_i^0}) \left[ \left( 16\pi b_i \mathcal{T}_{jj'}(z^0) \right. \right. \\
 & \left. \left. + \frac{4}{3}b_i \mathcal{T}_{kj'}(z^0)b_j^\perp b_k^\perp + \frac{1}{3}\delta_{jj'}b_i \mathcal{T}_{kl}(z^0)b_k^\perp b_l^\perp - \frac{1}{8\pi}\mathcal{T}_{kl}(\frac{\partial}{\partial z_i^0})b_j^\perp b_{j'}^\perp b_k^\perp b_l^\perp \right) P^N \right] (0,0) \\
 & = \frac{1}{28\pi^2} \left[ -\frac{64\pi}{3}|\mathcal{T}_{jj'}(\frac{\partial}{\partial \bar{z}_i^0})|^2 + \frac{8\pi}{3}\mathcal{T}_{jj'}(\frac{\partial}{\partial \bar{z}_i^0})\mathcal{T}_{kk}(\frac{\partial}{\partial z_i^0}) \right. \\
 & \quad \left. - 2\pi \left( 2|\mathcal{T}_{jj'}(\frac{\partial}{\partial \bar{z}_i^0})|^2 + \mathcal{T}_{jj'}(\frac{\partial}{\partial \bar{z}_i^0})\mathcal{T}_{kk}(\frac{\partial}{\partial z_i^0}) \right) \right] P^N(0,0) \\
 & = \frac{1}{28 \cdot 3\pi} \left[ -76|\mathcal{T}_{jj'}(\frac{\partial}{\partial \bar{z}_i^0})|^2 + 2\mathcal{T}_{jj'}(\frac{\partial}{\partial \bar{z}_i^0})\mathcal{T}_{kk}(\frac{\partial}{\partial z_i^0}) \right] P^N(0,0).
 \end{aligned}$$

By (3.9), (3.54), (5.140), we get

$$(5.146) \quad \mathcal{I}_3P^N = -\frac{\sqrt{-1}}{8\pi}\tilde{\mathcal{T}}_{ijj'} \left[ b_j^\perp b_{j'}^\perp \mathcal{F}_1(e_i^\perp) + \mathcal{T}_{ilm}b_j^\perp b_{j'}^\perp \frac{b_l^\perp b_m^\perp}{16\pi} + \frac{1}{2}\mathcal{T}_{ilj'}b_j^\perp b_l^\perp \right] P^N.$$

By (5.5e), (5.14), (5.135), (5.146) and a similar equation as (5.152) for  $\tilde{\mathcal{T}}_{ijj'}\mathcal{T}_{kli}b_j^\perp b_{j'}^\perp b_k^\perp b_l^\perp$ , we get

$$(5.147) \quad \left( (\mathcal{L}_2^0)^{-1}P^{N^\perp}\mathcal{I}_3P^N \right) (0,0) = \frac{\sqrt{-1}}{64\pi}\tilde{\mathcal{T}}_{ijj'}\mathcal{T}_{ijj'}P^N(0,0) = 0,$$

as  $\tilde{\mathcal{T}}_{ijj'}$  is anti-symmetric on  $i, j$  and  $\mathcal{T}_{ijj'}$  is symmetric on  $i, j$ .

By Theorem 3.1, (3.9), (3.54) and (5.135),

$$\begin{aligned}
 (5.148) \quad & \left( (\mathcal{L}_2^0)^{-1}P^{N^\perp}\mathcal{F}_1(e_j^\perp)(b_j^{\perp+} + b_j^\perp)\mathcal{F}_1(e_k^\perp)\frac{b_k^\perp}{4\pi}P^N \right) (0,0) \\
 & = \frac{1}{32\pi^2} \left( \mathcal{F}_1(e_j^\perp)^2(b_j^\perp)^2P^N \right) (0,0) = -\frac{1}{8\pi} \sum_j \mathcal{F}_1(e_j^\perp)^2P^N(0,0).
 \end{aligned}$$

Recall that  $\mathcal{T}_{klm}$  is symmetric on  $k, l, m$ .

By Theorem 3.1, (3.9), (3.54), (5.83) and (5.135),

$$\begin{aligned}
 (5.149) \quad & \left\{ (\mathcal{L}_2^0)^{-1}P^{N^\perp} \left( \mathcal{F}_1(e_j^\perp)(b_j^{\perp+} + b_j^\perp)\mathcal{T}_{klm}\frac{b_m^\perp b_l^\perp b_k^\perp}{192\pi^2} + \mathcal{T}_{ijj'}\frac{B_{ijj'}^\perp}{64\pi^2}\mathcal{F}_1(e_k^\perp)b_k^\perp \right) P^N \right\} (0,0) \\
 & = \left\{ (\mathcal{L}_2^0)^{-1}P^{N^\perp}\mathcal{F}_1(e_j^\perp) \left( b_j^\perp \mathcal{T}_{klm}\frac{b_k^\perp b_l^\perp b_m^\perp}{48\pi^2} + \mathcal{T}_{jlm}\frac{b_l^\perp b_m^\perp}{4\pi} \right) P^N \right\} (0,0) \\
 & = \frac{1}{32\pi^2} \left\{ \mathcal{F}_1(e_j^\perp) \left( \mathcal{T}_{klm}\frac{b_j^\perp b_k^\perp b_l^\perp b_m^\perp}{24\pi} + \mathcal{T}_{jlm}b_l^\perp b_m^\perp \right) P^N \right\} (0,0) \\
 & = \frac{1}{32\pi^2} \left\{ \mathcal{F}_1(e_j^\perp) \left( \sum_{l \neq j} \mathcal{T}_{jll}\frac{(b_j^\perp)^2(b_l^\perp)^2}{8\pi} + \mathcal{T}_{jjj}\frac{(b_j^\perp)^4}{24\pi} + \mathcal{T}_{jll}(b_l^\perp)^2 \right) P^N \right\} (0,0) \\
 & = -\frac{1}{16\pi}\mathcal{F}_1(e_j^\perp)\mathcal{T}_{jll}P^N(0,0).
 \end{aligned}$$

As  $\mathcal{T}_{klm}$  is symmetric on  $k, l, m$ , we know that

$$(5.150) \quad \begin{aligned} \mathcal{T}_{klm}^2 &= 6 \sum_{k < l < m} \mathcal{T}_{klm}^2 + 3 \sum_{k \neq m} \mathcal{T}_{kkm}^2 + \mathcal{T}_{mmm}^2, \\ \mathcal{T}_{kkm} \mathcal{T}_{llm} &= \sum_{k \neq l \neq m \neq k} \mathcal{T}_{kkm} \mathcal{T}_{llm} + \sum_{k \neq m} (2\mathcal{T}_{kkm} \mathcal{T}_{mmm} + \mathcal{T}_{kkm}^2) + \mathcal{T}_{mmm}^2. \end{aligned}$$

From (5.135), (5.150),

$$(5.151) \quad \begin{aligned} \left( \mathcal{T}_{ijj'} \mathcal{T}_{klm} b_i^\perp b_j^\perp b_{j'}^\perp b_k^\perp b_l^\perp b_m^\perp P^N \right) (0, 0) &= \left\{ \left( 36 \sum_{k < l < m} \mathcal{T}_{klm}^2 (b_k^\perp)^2 (b_l^\perp)^2 (b_m^\perp)^2 \right. \right. \\ &+ 9 \sum_{k \neq l \neq m \neq k} \mathcal{T}_{kkm} \mathcal{T}_{llm} (b_k^\perp)^2 (b_l^\perp)^2 (b_m^\perp)^2 + 6 \sum_{k \neq m} \mathcal{T}_{kkm} \mathcal{T}_{mmm} (b_k^\perp)^2 (b_m^\perp)^4 \\ &+ 9 \sum_{k \neq m} \mathcal{T}_{mmk} \mathcal{T}_{mmk} (b_k^\perp)^2 (b_m^\perp)^4 + \mathcal{T}_{mmm}^2 (b_m^\perp)^6 \Big) P^N \Big\} (0, 0) \\ &= (-4\pi)^3 \left( 36 \sum_{k < l < m} \mathcal{T}_{klm}^2 + 9 \sum_{k \neq l \neq m \neq k} \mathcal{T}_{kkm} \mathcal{T}_{llm} \right. \\ &+ 3 \sum_{k \neq m} (6\mathcal{T}_{kkm} \mathcal{T}_{mmm} + 9\mathcal{T}_{mmk} \mathcal{T}_{mmk}) + 15\mathcal{T}_{mmm}^2 \Big) P^N(0, 0) \\ &= (-4\pi)^3 \cdot 3 \left( 2\mathcal{T}_{klm}^2 + 3\mathcal{T}_{kkm} \mathcal{T}_{llm} \right) P^N(0, 0). \end{aligned}$$

By (5.135),

$$(5.152) \quad \begin{aligned} \left( \mathcal{T}_{ijm} \mathcal{T}_{klm} b_i^\perp b_j^\perp b_k^\perp b_l^\perp P^N \right) (0, 0) &= \left\{ \left[ \sum_{k \neq l} \left( 2\mathcal{T}_{klm}^2 + \mathcal{T}_{kkm} \mathcal{T}_{llm} \right) (b_k^\perp)^2 (b_l^\perp)^2 + \mathcal{T}_{llm}^2 (b_l^\perp)^4 \right] P^N \right\} (0, 0) \\ &= (4\pi)^2 \left( \sum_{k \neq l} (2\mathcal{T}_{klm}^2 + \mathcal{T}_{kkm} \mathcal{T}_{llm}) + 3\mathcal{T}_{llm}^2 \right) P^N(0, 0) \\ &= (4\pi)^2 (2\mathcal{T}_{klm}^2 + \mathcal{T}_{kkm} \mathcal{T}_{llm}) P^N(0, 0). \end{aligned}$$

By (3.9), (3.54) and (5.83), we have also

$$(5.153) \quad \begin{aligned} P^{N^\perp} \mathcal{T}_{ijj'} B_{ijj'}^\perp \mathcal{T}_{klm} b_k^\perp b_l^\perp b_m^\perp P^N &= \left( \mathcal{T}_{ijj'} \mathcal{T}_{klm} b_i^\perp b_j^\perp b_{j'}^\perp b_k^\perp b_l^\perp b_m^\perp \right. \\ &+ 36\pi \mathcal{T}_{ijm} \mathcal{T}_{klm} b_i^\perp b_j^\perp b_k^\perp b_l^\perp + 36\pi \cdot 8\pi \mathcal{T}_{ilm} \mathcal{T}_{klm} b_i^\perp b_k^\perp \Big) P^N. \end{aligned}$$

Thus from Theorem 3.1, (5.151)-(5.153),

$$\begin{aligned}
(5.154) \quad & \left\{ \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} \frac{1}{16\pi} \mathcal{T}_{ijj'} B_{ijj'}^\perp \mathcal{T}_{klm} \frac{b_k^\perp b_l^\perp b_m^\perp}{192\pi^2} \right) P^N \right\} (0, 0) \\
&= \frac{1}{2^{10} \cdot 3\pi^3} \left\{ \left( \frac{1}{24\pi} \mathcal{T}_{ijj'} \mathcal{T}_{klm} b_i^\perp b_j^\perp b_{j'}^\perp b_k^\perp b_l^\perp b_m^\perp + \frac{9}{4} \mathcal{T}_{ijm} \mathcal{T}_{klm} b_i^\perp b_j^\perp b_k^\perp b_l^\perp \right. \right. \\
&\quad \left. \left. + 36\pi \mathcal{T}_{ilm} \mathcal{T}_{klm} b_i^\perp b_k^\perp \right) P^N \right\} (0, 0) \\
&= \frac{1}{2^{10} \cdot 3\pi} \left\{ -8(2\mathcal{T}_{klm}^2 + 3\mathcal{T}_{kkm} \mathcal{T}_{llm}) + 36(2\mathcal{T}_{klm}^2 + \mathcal{T}_{kkm} \mathcal{T}_{llm}) - 144\mathcal{T}_{klm}^2 \right\} P^N(0, 0) \\
&= \frac{1}{2^8 \cdot 3\pi} (-22\mathcal{T}_{klm}^2 + 3\mathcal{T}_{kkm} \mathcal{T}_{llm}) P^N(0, 0).
\end{aligned}$$

From (5.141), (5.143), (5.145), (5.147), (5.148), (5.149) and (5.154), we get

$$\begin{aligned}
(5.155) \quad \Psi_{1,1}(0) &= \left\{ \frac{1}{2^8 \cdot 3\pi} \left[ -76|\mathcal{T}_{jj'}(\frac{\partial}{\partial \bar{z}_i^0})|^2 + 2\mathcal{T}_{jj'}(\frac{\partial}{\partial \bar{z}_i^0}) \mathcal{T}_{kk}(\frac{\partial}{\partial \bar{z}_i^0}) - 22\mathcal{T}_{klm}^2 + 3\mathcal{T}_{kkm} \mathcal{T}_{llm} \right] \right. \\
&\quad \left. - \frac{1}{24\pi} \mathcal{T}_{jj'}(\frac{\partial}{\partial \bar{z}_i^0}) \mathcal{T}_{kk}(\frac{\partial}{\partial \bar{z}_i^0}) - \frac{1}{8\pi} \sum_j \mathcal{F}_1(e_j^\perp)^2 - \frac{1}{16\pi} \mathcal{F}_1(e_j^\perp) \mathcal{T}_{jll} \right\} P^N(0, 0).
\end{aligned}$$

From (5.155) we get (5.139).  $\square$

Recall that  $B(Z, e_l^\perp)$  was defined in (5.23).

**Lemma 5.13.** *The following identity holds,*

$$\begin{aligned}
(5.156) \quad \frac{\sqrt{-1}}{\pi} B(Z, e_l^\perp) &= \frac{1}{2} \langle R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0) e_l^\perp, J\mathcal{R}^0 \rangle - \frac{5}{4} \langle \nabla_{\mathcal{R}}^{TY}(T(e_k, e_l^\perp)), J\mathcal{R}^\perp \rangle Z_k \\
&\quad + \frac{1}{2} \left\langle \frac{1}{3} R^{TB}(\mathcal{R}^\perp, e_l^\perp) \mathcal{R}^\perp + \nabla_{\mathcal{R}^0}^{TXG}(A(e_k^0) e_l^\perp) Z_k^0, J\mathcal{R}^0 \right\rangle \\
&\quad + \frac{1}{8} \langle T(\mathcal{R}^0, e_j^0), J e_l^\perp \rangle \langle T(\mathcal{R}^\perp - \mathcal{R}^0, J e_j^0), J\mathcal{R}^\perp \rangle \\
&\quad + \frac{1}{4} \langle T(\mathcal{R}^\perp, e_j^0), J e_l^\perp \rangle \langle T(\mathcal{R}^0, J e_j^0), J\mathcal{R}^\perp \rangle \\
&\quad + \frac{1}{8} \langle T(\mathcal{R}^0, J\mathcal{R}^0), T(\mathcal{R}^\perp, e_l^\perp) \rangle - \frac{1}{8} \langle T(\mathcal{R}, e_l^\perp), T(\mathcal{R}^\perp, J\mathcal{R}^0) \rangle \\
&\quad + \frac{1}{8} \langle T(\mathcal{R}^\perp, JT(\mathcal{R}^0, J\mathcal{R}^0)), J e_l^\perp \rangle + \frac{1}{2} \langle T(\mathcal{R}^\perp, J\mathcal{R}^\perp), T(\mathcal{R}, e_l^\perp) \rangle.
\end{aligned}$$

*Proof.* By (5.33), (5.54) and  $A(\mathcal{R}^0)A(\mathcal{R}^0)e_l^\perp \in N_G$ , as  $A$  exchanges  $TX_G$  and  $N_G$ , we get

$$\begin{aligned}
(5.157) \quad \langle J\mathcal{R}, (\nabla^{TX} \nabla^{TX} e_l^{\perp, H})_{(\mathcal{R}, \mathcal{R})} \rangle &= -\frac{1}{2} \langle J\mathcal{R}, T(\mathcal{R}, \nabla_{\mathcal{R}}^{TB} e_l^\perp) + \nabla_{\mathcal{R}}^{TX}(T(e_i^H, e_l^\perp)) Z_i \rangle \\
&\quad + \left\langle J\mathcal{R}^0, \frac{1}{3} R^{TB}(\mathcal{R}^\perp, e_l^\perp) \mathcal{R}^\perp + R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0) e_l^\perp + \nabla_{\mathcal{R}^0}^{TXG}(A(e_k^0) e_l^\perp) Z_k^0 \right\rangle.
\end{aligned}$$

By (1.8), (5.13), (5.53), we have at  $x_0$ ,

$$(5.158) \quad -\frac{1}{2} \langle J\mathcal{R}^\perp, T(\mathcal{R}, \nabla_{\mathcal{R}}^{TB} e_l^\perp) \rangle = \frac{1}{4} \langle J e_l^\perp, T(\mathcal{R}^0, e_j^0) \rangle \langle J\mathcal{R}^\perp, T(\mathcal{R}, J e_j^0) \rangle, \\ -\frac{1}{2} \langle J\mathcal{R}^0, \nabla_{\mathcal{R}}^{TX} (T(e_i^H, e_l^{\perp, H})) Z_i \rangle = -\frac{1}{4} \langle T(\mathcal{R}, e_l^\perp), T(\mathcal{R}, J\mathcal{R}^0) \rangle.$$

By (5.5a), (5.5d), (5.13), (5.53), (5.54) and  $\nabla_{\mathcal{R}}^{TX} (T(e_i^H, e_k^H)) Z_i Z_k = 0$ , we have

$$(5.159) \quad \langle J(\nabla^{TX} \nabla^{TX} e_k^H)_{(\mathcal{R}, \mathcal{R})}, e_l^\perp \rangle Z_k = \frac{1}{2} \langle T(\mathcal{R}, \nabla_{\mathcal{R}}^{TB} e_k), J e_l^\perp \rangle Z_k \\ = \frac{1}{2} \langle T(\mathcal{R}, 2A(\mathcal{R}^0)\mathcal{R}^\perp + A(\mathcal{R}^0)\mathcal{R}^0), J e_l^\perp \rangle \\ = \frac{1}{2} \langle T(\mathcal{R}, e_j^0), J e_l^\perp \rangle \langle T(\mathcal{R}^0, J e_j^0), J\mathcal{R}^\perp \rangle \\ - \frac{1}{4} \langle T(\mathcal{R}^0, e_l^\perp), T(\mathcal{R}^0, J\mathcal{R}^0) \rangle + \frac{1}{4} \langle T(\mathcal{R}^\perp, JT(\mathcal{R}^0, J\mathcal{R}^0)), J e_l^\perp \rangle.$$

From (3.40), (5.5a), (5.13), (5.53) and the fact that  $A$  exchanges  $TX_G$  and  $N_G$ , we get

$$(5.160) \quad \langle J\nabla_{\mathcal{R}}^{TX} e_k^H, \nabla_{\mathcal{R}}^{TX} e_l^{\perp, H} \rangle Z_k = \left\langle J\nabla_{\mathcal{R}}^{TB} e_k, A(\mathcal{R}^0) e_l^\perp - \frac{1}{2} T(\mathcal{R}, e_l^\perp) \right\rangle Z_k \\ = \left\langle JA(\mathcal{R}^0)\mathcal{R}^0, -\frac{1}{2} T(\mathcal{R}, e_l^\perp) \right\rangle + 2 \langle JA(\mathcal{R}^0)\mathcal{R}^\perp, A(\mathcal{R}^0) e_l^\perp \rangle, \\ = \frac{1}{4} \langle T(\mathcal{R}^0, J\mathcal{R}^0), T(\mathcal{R}, e_l^\perp) \rangle - \frac{1}{2} \langle J e_l^\perp, T(\mathcal{R}^0, e_j^0) \rangle \langle J\mathcal{R}^\perp, T(\mathcal{R}^0, J e_j^0) \rangle.$$

From (5.51), (5.52), (5.61), (5.157)-(5.160), we get

$$(5.161) \quad \frac{\sqrt{-1}}{\pi} B(Z, e_l^\perp) = \frac{1}{8} \langle J e_l^\perp, T(\mathcal{R}^0, e_j^0) \rangle \langle J\mathcal{R}^\perp, T(\mathcal{R}, J e_j^0) \rangle \\ - \frac{1}{4} \langle J\mathcal{R}^\perp, \nabla_{\mathcal{R}}^{TY} (T(e_i, e_l^\perp)) Z_i \rangle - \frac{1}{8} \langle T(\mathcal{R}, e_l^\perp), T(\mathcal{R}, J\mathcal{R}^0) \rangle \\ + \frac{1}{2} \left\langle J\mathcal{R}^0, \frac{1}{3} R^{TB}(\mathcal{R}^\perp, e_l^\perp) \mathcal{R}^\perp + R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0) e_l^\perp + \nabla_{\mathcal{R}^0}^{TX_G} (A(e_k^0) e_l^\perp) Z_k^0 \right\rangle \\ + \frac{1}{4} \langle T(\mathcal{R}, e_j^0), J e_l^\perp \rangle \langle T(\mathcal{R}^0, J e_j^0), J\mathcal{R}^\perp \rangle - \frac{1}{8} \langle T(\mathcal{R}^0, e_l^\perp), T(\mathcal{R}^0, J\mathcal{R}^0) \rangle \\ + \frac{1}{8} \langle T(\mathcal{R}^\perp, JT(\mathcal{R}^0, J\mathcal{R}^0)), J e_l^\perp \rangle + \frac{1}{4} \langle T(\mathcal{R}^0, J\mathcal{R}^0), T(\mathcal{R}, e_l^\perp) \rangle \\ - \frac{1}{2} \langle J e_l^\perp, T(\mathcal{R}^0, e_j^0) \rangle \langle J\mathcal{R}^\perp, T(\mathcal{R}^0, J e_j^0) \rangle \\ + \frac{1}{2} \langle T(\mathcal{R}^\perp, J\mathcal{R}^\perp), T(\mathcal{R}, e_l^\perp) \rangle - \langle \nabla_{\mathcal{R}}^{TY} (T(e_k, e_l^\perp)), J\mathcal{R}^\perp \rangle Z_k.$$

From (5.161) we get (5.156). □

Now we need to compute the contribution from  $-(\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{O}_2 P^N$ .

**Lemma 5.14.** *The following identity holds,*

$$\begin{aligned}
(5.162) \quad \Psi_{1,2}(0) = & \left\{ \frac{1}{16\pi} r_{x_0}^{X_G} + \frac{1}{2\pi} R^{E_G} \left( \frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0} \right) + \frac{1}{2\pi} \Delta_{X_G} \log h + \frac{29}{2^5 \cdot 3\pi} |T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})|^2 \right. \\
& + \frac{\sqrt{-1}}{16\pi} \left\langle T(e_k^\perp, J e_k^\perp), T\left(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0}\right) \right\rangle + \frac{1}{4\pi} \left| T\left(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0}\right) \right|^2 + \frac{1}{32\pi} \left| \sum_j \mathcal{T}_{jj} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) \right|^2 \\
& - \frac{1}{2^6 \pi} \left\langle (\nabla^{TY} \dot{g}^{TY})_{(e_j^\perp, e_j^\perp)} J e_k^\perp, J e_k^\perp \right\rangle - \frac{1}{2^5 \pi} \left\langle (\nabla^{TY} \dot{g}^{TY})_{(e_j^\perp, e_k^\perp)} J e_j^\perp, J e_k^\perp \right\rangle \\
& + \frac{1}{2^7 \pi} \tilde{\mathcal{T}}_{ijk} (\tilde{\mathcal{T}}_{kji} + \tilde{\mathcal{T}}_{ijk}) + \frac{7}{2^8 \pi} (2\mathcal{T}_{jkm}^2 + \mathcal{T}_{jkm} \mathcal{T}_{kkm}) \\
& \left. - \frac{\sqrt{-1}}{16\pi} \left( \left\langle T(e_j^\perp, J e_j^\perp), \tilde{\mu}^E \right\rangle - 2 \left\langle J e_j^\perp, \nabla_{e_j^\perp}^{TY} \tilde{\mu}^E \right\rangle \right) \right\} P^N(0, 0).
\end{aligned}$$

*Proof.* From Theorem 3.1, (3.9), (3.54), (5.15) and (5.135),

$$\begin{aligned}
(5.163) \quad 4\pi \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} Z_k^\perp Z_l^\perp P^N \right) (0, 0) &= \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} Z_l^\perp b_k^\perp P^N \right) (0, 0) \\
&= \left( (\mathcal{L}_2^0)^{-1} b_k^\perp Z_l^\perp P^N \right) (0, 0) = \left( \frac{b_k^\perp b_l^\perp}{32\pi^2} P^N \right) (0, 0) = -\frac{\delta_{kl}}{8\pi} P^N(0, 0).
\end{aligned}$$

Set

$$(5.164) \quad \mathcal{I}_4 = - \left\{ (\mathcal{L}_2^0)^{-1} P^{N^\perp} \left( \frac{\partial}{\partial z_j^0} \left( B(Z, \frac{\partial}{\partial \bar{z}_j^0}) \right) - \frac{\partial}{\partial \bar{z}_j^0} \left( B(Z, \frac{\partial}{\partial z_j^0}) \right) \right) P^N \right\} (0, 0).$$

At first, if  $Q$  is a monomial on  $b_i, b_i^+, b_j^\perp, b_j^{\perp+}, Z_i$  and the total degree of  $b_i, b_i^+, Z_i^0$  or  $b_j^\perp, b_j^{\perp+}, Z_j^\perp$  is odd, then by Theorem 3.1,

$$(5.165) \quad \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} Q P^N \right) (0, 0) = 0.$$

By (5.165), only the monomials of  $B(Z, e_l^0)$  with odd degree on  $Z^0$  have contributions for  $\mathcal{I}_4$ .

If we denote by  $\tilde{B}_Z(e_l^0)$  the odd degree component on  $Z^0$  of the difference of  $B(Z, e_l^0)$  and of the sum the the first two and the last terms of  $B(Z, e_l^0)$  in (5.45b), then by (5.45b) we know that  $\tilde{B}_Z(e_l^0)$  is a linear function on  $Z^0$  and  $\frac{\partial}{\partial z_j^0} \left( \tilde{B}_Z(\frac{\partial}{\partial \bar{z}_j^0}) \right)$  and  $-\frac{\partial}{\partial \bar{z}_j^0} \left( \tilde{B}_Z(\frac{\partial}{\partial z_j^0}) \right)$  are equal.

Moreover, by (5.5e), (5.163), we know the contribution of the last term of  $B(Z, e_l^0)$  in (5.45b) is zero in  $\mathcal{I}_4$ .

Thus by Remark 5.2, (5.45b) and (5.164),

$$\begin{aligned}
(5.166) \quad \mathcal{I}_4 = & \pi\sqrt{-1} \left\{ (\mathcal{L}_2^0)^{-1} P^{N^\perp} \left[ \frac{1}{6} \frac{\partial}{\partial z_j^0} \left\langle R^{TX_G}(\mathcal{R}^0, J\mathcal{R}^0) \mathcal{R}^0, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle \right. \right. \\
& - \frac{1}{6} \frac{\partial}{\partial \bar{z}_j^0} \left\langle R^{TX_G}(\mathcal{R}^0, J\mathcal{R}^0) \mathcal{R}^0, \frac{\partial}{\partial z_j^0} \right\rangle \\
& - \frac{5}{4} \left\langle J\mathcal{R}^\perp, 2\nabla_{\mathcal{R}^\perp}^{TY} \left( T\left(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0}\right) \right) + \nabla_{\frac{\partial}{\partial z_j^0}}^{TY} \left( T(e_i^\perp, \frac{\partial}{\partial \bar{z}_j^0}) \right) Z_i^\perp - \nabla_{\frac{\partial}{\partial \bar{z}_j^0}}^{TY} \left( T(e_i^\perp, \frac{\partial}{\partial z_j^0}) \right) Z_i^\perp \right\rangle \\
& + 3\sqrt{-1} \left\langle R^{TB}(\mathcal{R}^\perp, \frac{\partial}{\partial \bar{z}_j^0}) \mathcal{R}^\perp, \frac{\partial}{\partial z_j^0} \right\rangle - \frac{3\sqrt{-1}}{4} \left\langle J\mathcal{R}^\perp, T\left(\frac{\partial}{\partial z_j^0}, e_i^0\right) \right\rangle \left\langle J\mathcal{R}^\perp, T(e_i^0, \frac{\partial}{\partial \bar{z}_j^0}) \right\rangle \\
& \left. - \frac{\sqrt{-1}}{4} \left\langle T(\mathcal{R}^\perp, \frac{\partial}{\partial \bar{z}_j^0}), T(\mathcal{R}^\perp, \frac{\partial}{\partial z_j^0}) \right\rangle + \left\langle T(\mathcal{R}^\perp, J\mathcal{R}^\perp), T\left(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0}\right) \right\rangle \right] P^N \right\} (0, 0).
\end{aligned}$$

By (5.92), (5.107a), (5.163) and (5.166), comparing with (5.103) and (5.104), we get

$$\begin{aligned}
(5.167) \quad \mathcal{I}_4 = & \left\{ -\frac{1}{6\pi} \left\langle R^{TX_G} \left( \frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_i^0} \right) \frac{\partial}{\partial z_j^0} + R^{TX_G} \left( \frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_i^0} \right) \frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle \right. \\
& + \frac{5\sqrt{-1}}{27\pi} \left\langle J e_k^\perp, 2\nabla_{e_k^\perp}^{TY} \left( T\left(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0}\right) \right) + \nabla_{\frac{\partial}{\partial z_j^0}}^{TY} \left( T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) \right) - \nabla_{\frac{\partial}{\partial \bar{z}_j^0}}^{TY} \left( T(e_k^\perp, \frac{\partial}{\partial z_j^0}) \right) \right\rangle \\
& + \frac{3}{32\pi} \left\langle R^{TB}(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) e_k^\perp, \frac{\partial}{\partial z_j^0} \right\rangle + \frac{3}{64\pi} |T(\frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0})|^2 \\
& \left. - \frac{1}{27\pi} |T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})|^2 - \frac{\sqrt{-1}}{32\pi} \left\langle T(e_k^\perp, J e_k^\perp), T\left(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0}\right) \right\rangle \right\} P^N(0, 0).
\end{aligned}$$

By (3.9), (3.54) and (5.83),

$$\begin{aligned}
(5.168) \quad & (z_i^0 \bar{z}_j^0 P^N)(Z, 0) = (z_i^0 \frac{b_j}{2\pi} P^N)(Z, 0) = \frac{1}{2\pi} ((b_j z_i^0 + 2\delta_{ij}) P^N)(Z, 0), \\
& Z_k^\perp Z_l^\perp P^N = \frac{1}{16\pi^2} (b_k^\perp b_l^\perp + 4\pi\delta_{kl}) P^N, \\
& (4\pi)^4 (Z_k^\perp)^4 P^N = ((b_k^\perp)^4 + 24\pi(b_k^\perp)^2 + 3 \cdot (4\pi)^2) P^N.
\end{aligned}$$

From Theorem 3.1, (3.9), (3.54), (5.92), (5.135), (5.142) and (5.168),

$$\begin{aligned}
 (P^{N^\perp} Z_k^\perp Z_l^\perp P^N)(0, 0) &= \frac{1}{16\pi^2} (b_k^\perp b_l^\perp P^N)(0, 0) = -\frac{\delta_{kl}}{4\pi}, \\
 (5.169) \quad & \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} b_j z_i^0 Z_k^\perp Z_l^\perp P^N \right) (0, 0) \\
 &= \frac{1}{16\pi^2} \left\{ \left( \frac{1}{12\pi} b_k^\perp b_l^\perp b_j z_i^0 + \delta_{kl} b_j z_i^0 \right) P^N \right\} (0, 0) = -\frac{1}{12\pi^2} \delta_{ij} \delta_{kl} P^N(0, 0), \\
 & \left( (\mathcal{L}_2^0)^{-1} b_l^\perp Z_k^\perp z_i^0 \bar{z}_j^0 P^N \right) (0, 0) = \frac{1}{8\pi^2} \left\{ b_l^\perp b_k^\perp \left( \frac{b_j}{12\pi} z_i^0 + \frac{2}{8\pi} \delta_{ij} \right) P^N \right\} (0, 0) \\
 &= -\frac{1}{24\pi^2} \delta_{ij} \delta_{kl} P^N(0, 0), \\
 & \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} Z_l^\perp Z_k^\perp z_i^0 \bar{z}_j^0 P^N \right) (0, 0) \\
 &= \frac{1}{4\pi} \left\{ (\mathcal{L}_2^0)^{-1} P^{N^\perp} (b_l^\perp Z_k^\perp + \delta_{kl}) z_i^0 \bar{z}_j^0 P^N \right\} (0, 0) = \frac{-7}{96\pi^3} \delta_{ij} \delta_{kl} P^N(0, 0).
 \end{aligned}$$

By (5.5e), (5.106), (5.107a), (5.169) and comparing with (5.108), we get

$$\begin{aligned}
 (5.170) \quad & - \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} b_j B(Z, \frac{\partial}{\partial \bar{z}_j^0}) P^N \right) (0, 0) \\
 &= \left\{ -\frac{1}{12\pi} \left\langle R^{TX_G} \left( \frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_i^0} \right) \frac{\partial}{\partial z_i^0} + R^{TX_G} \left( \frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_i^0} \right) \frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle \right. \\
 &\quad + \frac{5\sqrt{-1}}{48\pi} \left\langle \nabla_{\frac{\partial}{\partial z_j^0}}^{TY} \left( T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) \right) + \nabla_{e_k^\perp}^{TY} \left( T(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0}) \right), J e_k^\perp \right\rangle \\
 &\quad + \frac{1}{8\pi} \left\langle R^{TB} \left( e_k^\perp, \frac{\partial}{\partial z_j^0} \right) e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle + \frac{1}{16\pi} \left| T \left( \frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0} \right) \right|^2 \\
 &\quad \left. - \frac{1}{96\pi} \left| T \left( e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0} \right) \right|^2 - \frac{\sqrt{-1}}{24\pi} \left\langle T(e_k^\perp, J e_k^\perp), T \left( \frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0} \right) \right\rangle \right\} P(0, 0).
 \end{aligned}$$

From (5.156) and (5.165),

$$\begin{aligned}
 (5.171) \quad & \left( (\mathcal{L}_2^0)^{-1} b_l^\perp B(Z, e_l^\perp) P^N \right) (0, 0) = -\pi\sqrt{-1} \left\{ (\mathcal{L}_2^0)^{-1} b_l^\perp \right. \\
 & \quad \left[ \frac{1}{2} \left\langle R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0) e_l^\perp, J \mathcal{R}^0 \right\rangle - \frac{5}{4} \left\langle \nabla_{\mathcal{R}^\perp}^{TY} (T(e_k^\perp, e_l^\perp)), J \mathcal{R}^\perp \right\rangle Z_k^\perp \right. \\
 & \quad - \frac{5}{4} \left\langle \nabla_{\mathcal{R}^0}^{TY} (T(e_k^0, e_l^\perp)), J \mathcal{R}^\perp \right\rangle Z_k^0 - \frac{1}{8} \left\langle T(\mathcal{R}^0, e_j^0), J e_l^\perp \right\rangle \left\langle T(\mathcal{R}^0, J e_j^0), J \mathcal{R}^\perp \right\rangle \\
 & \quad + \frac{1}{8} \left\langle T(\mathcal{R}^0, J \mathcal{R}^0), T(\mathcal{R}^\perp, e_l^\perp) \right\rangle - \frac{1}{8} \left\langle T(\mathcal{R}^0, e_l^\perp), T(\mathcal{R}^\perp, J \mathcal{R}^0) \right\rangle \\
 & \quad \left. \left. + \frac{1}{8} \left\langle T(\mathcal{R}^\perp, J T(\mathcal{R}^0, J \mathcal{R}^0)), J e_l^\perp \right\rangle + \frac{1}{2} \left\langle T(\mathcal{R}^\perp, J \mathcal{R}^\perp), T(\mathcal{R}^\perp, e_l^\perp) \right\rangle \right] P^N \right\} (0, 0).
 \end{aligned}$$

As  $T$  is anti-symmetric, from (3.9), (3.54), we get

$$\begin{aligned}
 (5.172) \quad & b_l^\perp \left\langle \nabla_{\mathcal{R}^\perp}^{TY} (T(e_k^\perp, e_l^\perp)), J \mathcal{R}^\perp \right\rangle Z_k^\perp P^N = - \left( \frac{\partial}{\partial \bar{z}_l^\perp} \left\langle \nabla_{\mathcal{R}^\perp}^{TY} (T(e_k^\perp, e_l^\perp)), J \mathcal{R}^\perp \right\rangle \right) Z_k^\perp P^N, \\
 & b_l^\perp \left\langle T(\mathcal{R}^\perp, J \mathcal{R}^\perp), T(\mathcal{R}^\perp, e_l^\perp) \right\rangle P^N = - \left\langle T(\mathcal{R}^\perp, J e_l^\perp) + T(e_l^\perp, J \mathcal{R}^\perp), T(\mathcal{R}^\perp, e_l^\perp) \right\rangle P^N.
 \end{aligned}$$

From (5.5e), (5.123), (5.163), (5.169), (5.171), (5.172) and the anti-symmetric property of  $T$ , we get

$$\begin{aligned}
 (5.173) \quad & -\frac{1}{2} \left( (\mathcal{L}_2^0)^{-1} b_l^\perp B(Z, e_l^\perp) P^N \right) (0, 0) \\
 &= \frac{\sqrt{-1}}{2\pi} \left\{ -\frac{5}{2^7} \left( \left\langle \nabla_{e_k^\perp}^{TY} (T(e_k^\perp, e_l^\perp)), J e_l^\perp \right\rangle + \left\langle \nabla_{e_l^\perp}^{TY} (T(e_k^\perp, e_l^\perp)), J e_k^\perp \right\rangle \right) \right. \\
 &\quad + \frac{5}{96} \left\langle \nabla_{\frac{\partial}{\partial z_j^0}}^{TY} (T(\frac{\partial}{\partial \bar{z}_j^0}, e_l^\perp)) + \nabla_{\frac{\partial}{\partial \bar{z}_j^0}}^{TY} (T(\frac{\partial}{\partial z_j^0}, e_l^\perp)), J e_l^\perp \right\rangle \\
 &\quad \left. + \frac{1}{2^6} \left\langle T(e_k^\perp, J e_l^\perp) + T(e_l^\perp, J e_k^\perp), T(e_k^\perp, e_l^\perp) \right\rangle \right\} P^N(0, 0) = 0.
 \end{aligned}$$

By (5.101), (5.123), (5.164), (5.167), (5.170), (5.173) and since  $R^{TX_G}(\cdot, \cdot)$  is a  $(1, 1)$ -form, comparing with (5.104) and (5.108), we get

$$\begin{aligned}
 (5.174) \quad & - \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} I_1 P^N \right) (0, 0) = \left\{ -\frac{1}{2\pi} \left\langle R^{TX_G} \left( \frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_i^0} \right) \frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle \right. \\
 &\quad + \frac{7}{6} \left[ \frac{5\sqrt{-1}}{2^5 \pi} \left\langle J e_k^\perp, \nabla_{e_k^\perp}^{TY} (T(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0})) + \nabla_{\frac{\partial}{\partial \bar{z}_j^0}}^{TY} (T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})) \right\rangle \right. \\
 &\quad \left. + \frac{3}{16\pi} \left\langle R^{TB} (e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle + \frac{3}{32\pi} |T(\frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0})|^2 \right. \\
 &\quad \left. \left. - \frac{1}{2^6 \pi} |T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})|^2 - \frac{\sqrt{-1}}{16\pi} \left\langle T(e_k^\perp, J e_k^\perp), T(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0}) \right\rangle \right] \right\} P^N(0, 0).
 \end{aligned}$$

Recall that from (3.6), (5.5a), (5.5b) and (5.13),

$$\begin{aligned}
 (5.175) \quad & |A(e_i^0) e_k^\perp|^2 = 4 |A(\frac{\partial}{\partial z_i^0}) e_k^\perp|^2 = |T(\frac{\partial}{\partial z_i^0}, J e_j^0)|^2 = 2 |T(\frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0})|^2, \\
 & \langle A(e_i^0) e_i^0, A(e_j^0) e_j^0 \rangle = 4 \left| \sum_i T(\frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_i^0}) \right|^2, \\
 & |A(e_i^0) e_j^0|^2 = \frac{1}{4} |T(e_i^0, J e_j^0)|^2 = |T(\frac{\partial}{\partial z_i^0}, J e_j^0)|^2 = 2 |T(\frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0})|^2.
 \end{aligned}$$

From (5.92), (5.110), (5.163), (5.175) and since  $R^{TX_G}(\cdot, \cdot)$  is a  $(1, 1)$ -form (comparing with (5.112b), (5.113)), we get

$$\begin{aligned}
 (5.176) \quad & - \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} I_2 P^N \right) (0, 0) = \left\{ \frac{4}{3\pi} \left\langle R^{TX_G} \left( \frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_i^0} \right) \frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle \right. \\
 &\quad \left. - \frac{1}{8\pi} \left\langle R^{TB} (e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle - \frac{1}{48\pi} \left\langle R^{TB} (e_k^\perp, e_j^\perp) e_k^\perp, e_j^\perp \right\rangle + \frac{3}{16\pi} |T(\frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0})|^2 \right\} P^N(0, 0).
 \end{aligned}$$



By (3.6), (3.54), (5.24), (5.82), (5.92), (5.111), (5.163), (5.175) and since  $R^{TXG}(\cdot, \cdot)$  is a  $(1, 1)$ -form (comparing with (5.112a)), we get

$$\begin{aligned}
(5.177) \quad & - \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} \langle \Gamma_{ii}(\mathcal{R}), e_l \rangle \nabla_{0, e_l} P^N \right) (0, 0) \\
& = \left\{ (\mathcal{L}_2^0)^{-1} P^{N^\perp} \left( \frac{2}{3} \left\langle R^{TXG}(\mathcal{R}^0, e_i^0) e_i^0, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle b_j \right. \right. \\
& \quad \left. \left. + \frac{1}{2} \langle R^{TB}(\mathcal{R}^\perp, e_i^0) e_i^0 + A(e_i^0) A(e_i^0) \mathcal{R}^\perp, e_k^\perp \rangle b_k^\perp \right) P^N \right\} (0, 0) \\
& = \left\{ -\frac{1}{3\pi} \left\langle R^{TXG}(\frac{\partial}{\partial z_j^0}, e_i^0) e_i^0, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle - \frac{1}{16\pi} \langle R^{TB}(e_k^\perp, e_i^0) e_i^0, e_k^\perp \rangle + \frac{1}{16\pi} |A(e_i^0) e_k^\perp|^2 \right\} P^N(0, 0) \\
& = \left\{ \left\langle -\frac{2}{3\pi} R^{TXG}(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_i^0}) \frac{\partial}{\partial z_i^0} + \frac{1}{4\pi} R^{TB}(e_k^\perp, \frac{\partial}{\partial z_j^0}) e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle + \frac{1}{8\pi} |T(\frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0})|^2 \right\} P^N(0, 0).
\end{aligned}$$

By  $\mathcal{L}_2^0 P^N = 0$ , (5.24), (5.92), (5.169), (5.175) and since  $R^{TXG}(\cdot, \cdot)$  is a  $(1, 1)$ -form (comparing with (5.114)), we get

$$\begin{aligned}
(5.178) \quad & - \left\{ (\mathcal{L}_2^0)^{-1} P^{N^\perp} \left[ \frac{1}{4} K_2(\mathcal{R}) - \frac{3}{8} \left( \sum_l \langle A(e_l^0) e_l^0, \mathcal{R}^\perp \rangle \right)^2, \mathcal{L}_2^0 \right] P^N \right\} (0, 0) \\
& = \left\{ P^{N^\perp} \left[ \frac{1}{4} K_2(\mathcal{R}) - \frac{3}{8} \left( \sum_l \langle A(e_l^0) e_l^0, \mathcal{R}^\perp \rangle \right)^2 \right] P^N \right\} (0, 0) \\
& = \frac{1}{4} \left\{ P^{N^\perp} \left[ \left\langle \frac{1}{3} R^{TXG}(\mathcal{R}^0, e_i^0) \mathcal{R}^0 + R^{TB}(\mathcal{R}^\perp, e_i^0) \mathcal{R}^\perp, e_i^0 \right\rangle \right. \right. \\
& \quad \left. \left. + \frac{1}{3} \langle R^{TB}(\mathcal{R}^\perp, e_i^\perp) \mathcal{R}^\perp, e_i^\perp \rangle + \frac{1}{2} \left( \sum_i \langle A(e_i^0) e_i^0, \mathcal{R}^\perp \rangle \right)^2 - |A(e_i^0) \mathcal{R}^\perp|^2 \right] P^N \right\} (0, 0) \\
& = \left( \frac{1}{6\pi} \left\langle R^{TXG}(\frac{\partial}{\partial z_j^0}, e_i^0) e_i^0, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle - \frac{1}{16\pi} \langle R^{TB}(e_k^\perp, e_i^0) e_k^\perp, e_i^0 \rangle \right. \\
& \quad \left. - \frac{1}{48\pi} \langle R^{TB}(e_k^\perp, e_i^\perp) e_k^\perp, e_i^\perp \rangle - \frac{1}{32\pi} \left| \sum_i A(e_i^0) e_i^0 \right|^2 + \frac{1}{16\pi} |A(e_i^0) e_k^\perp|^2 \right) P^N(0, 0), \\
& = \left( \frac{1}{3\pi} \left\langle R^{TXG}(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_i^0}) \frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle - \frac{1}{4\pi} \left\langle R^{TB}(e_k^\perp, \frac{\partial}{\partial z_j^0}) e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle \right. \\
& \quad \left. - \frac{1}{8\pi} \left| \sum_i T(\frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_i^0}) \right|^2 + \frac{1}{8\pi} |T(\frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0})|^2 - \frac{1}{48\pi} \langle R^{TB}(e_k^\perp, e_j^\perp) e_k^\perp, e_j^\perp \rangle \right) P^N(0, 0).
\end{aligned}$$

By (3.12), (3.54), (5.82), (5.92), (5.163) and (5.175),

$$\begin{aligned}
(5.179) \quad & - \left\{ (\mathcal{L}_2^0)^{-1} P^{N^\perp} \left( -\frac{1}{2} \langle A(e_l^0) e_l^0, \mathcal{R}^\perp \rangle \nabla_{A(e_k^0) e_k^0} + 2 \langle A(e_i^0) e_j^0, \mathcal{R}^\perp \rangle \nabla_{A(e_i^0) e_j^0} \right. \right. \\
& \quad \left. \left. + \frac{2}{3} \langle R^{TB}(\mathcal{R}^\perp, e_i^\perp) e_i^\perp, e_j \rangle \nabla_{0, e_j} \right) P^N \right\} (0, 0) \\
& = -\frac{1}{16\pi} \left( -\frac{1}{2} \left| \sum_l A(e_l^0) e_l^0 \right|^2 + 2 |A(e_i^0) e_j^0|^2 + \frac{2}{3} \langle R^{TB}(e_j^\perp, e_i^\perp) e_i^\perp, e_j^\perp \rangle \right) P^N(0, 0), \\
& = \left( \frac{1}{8\pi} \left| \sum_i T\left(\frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_i^0}\right) \right|^2 - \frac{1}{4\pi} \left| T\left(\frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0}\right) \right|^2 + \frac{1}{24\pi} \langle R^{TB}(e_k^\perp, e_j^\perp) e_k^\perp, e_j^\perp \rangle \right) P^N(0, 0), \\
& - \left\{ (\mathcal{L}_2^0)^{-1} P^{N^\perp} (-R^{EB}(\mathcal{R}, e_i)) \nabla_{0, e_i} P^N \right\} (0, 0) = \frac{1}{2\pi} R^{EB}\left(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0}\right) P^N(0, 0).
\end{aligned}$$

For  $F_{ij;kl} \in \mathbb{C}$ , from Theorem 3.1, (5.15), (5.135), (5.168) and comparing with (5.152), we get

$$\begin{aligned}
(5.180) \quad & \left\{ (\mathcal{L}_2^0)^{-1} P^{N^\perp} F_{ij;kl} Z_i^\perp Z_j^\perp Z_k^\perp Z_l^\perp P^N \right\} (0, 0) \\
& = \left\{ (\mathcal{L}_2^0)^{-1} P^{N^\perp} \left[ \sum_{j \neq k} (F_{jj;kk} + F_{kj;kj} + F_{kj;jk}) (Z_j^\perp)^2 (Z_k^\perp)^2 + F_{kk;kk} (Z_k^\perp)^4 \right] P^N \right\} (0, 0) \\
& = \frac{1}{2^8 \pi^4} \left\{ P^{N^\perp} \left[ \sum_{j \neq k} (F_{jj;kk} + F_{kj;kj} + F_{kj;jk}) \left( \frac{(b_j^\perp)^2 (b_k^\perp)^2}{16\pi} + \frac{1}{2} ((b_j^\perp)^2 + (b_k^\perp)^2) \right) \right. \right. \\
& \quad \left. \left. + F_{kk;kk} \left( \frac{(b_k^\perp)^4}{16\pi} + 3(b_k^\perp)^2 \right) \right] P^N \right\} (0, 0) \\
& = \frac{-3}{2^8 \pi^3} (F_{jj;kk} + F_{kj;kj} + F_{kj;jk}) P^N(0, 0).
\end{aligned}$$

By (5.45a), (5.74),

$$\begin{aligned}
(5.181) \quad & \frac{1}{9} \sum_i \left[ (\partial_{\mathcal{R}} R^{LB})_{x_0}(\mathcal{R}, e_i) \right]^2 = -\pi^2 \sum_i \langle JT(\mathcal{R}^\perp, e_i^0), \mathcal{R}^\perp \rangle^2 \\
& \quad - \pi^2 \sum_j \langle JT(\mathcal{R}, e_j^\perp), \mathcal{R}^\perp \rangle^2.
\end{aligned}$$

By (3.6), (5.14), (5.180) and  $\mathcal{T}_{kl}(e_i^0)$  is symmetric on  $k, l$ , we get

$$\begin{aligned}
(5.182) \quad & -\pi^2 \sum_i \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} \langle JT(\mathcal{R}^\perp, e_i^0), \mathcal{R}^\perp \rangle^2 P^N \right) (0, 0) \\
& = -\pi^2 \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{T}_{jj'}(e_i^0) \mathcal{T}_{kl}(e_i^0) Z_j^\perp Z_{j'}^\perp Z_k^\perp Z_l^\perp P^N \right) (0, 0) \\
& = \frac{3}{2^8 \pi} \left( 2 \mathcal{T}_{jk}(e_i^0)^2 + \mathcal{T}_{jj}(e_i^0) \mathcal{T}_{kk}(e_i^0) \right) P^N(0, 0) \\
& = \frac{3}{2^6 \pi} \left( 2 \left| T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) \right|^2 + \left| \sum_j \mathcal{T}_{jj}(\frac{\partial}{\partial \bar{z}_i^0}) \right|^2 \right) P^N(0, 0).
\end{aligned}$$

In the same way, by (5.5e), (5.14), (5.180), we get

$$(5.183) \quad -\pi^2 \sum_j \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} \langle JT(\mathcal{R}^\perp, e_j^\perp), \mathcal{R}^\perp \rangle^2 P^N \right) (0, 0) \\ = \frac{3}{2^8 \pi} \tilde{\mathcal{T}}_{ijk} (\tilde{\mathcal{T}}_{ijk} + \tilde{\mathcal{T}}_{kji}) P^N(0, 0).$$

By (5.14), (5.169),

$$(5.184) \quad -\pi^2 \sum_j \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} \langle JT(\mathcal{R}^0, e_j^\perp), \mathcal{R}^\perp \rangle^2 P^N \right) (0, 0) \\ = \frac{7}{48\pi} |\mathcal{T}_{jk}(\frac{\partial}{\partial \bar{z}_i^0})|^2 P^N(0, 0) = \frac{7}{48\pi} |T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})|^2 P^N(0, 0).$$

By (5.45a) and (5.115), the total degree of  $Z^0$ ,  $\nabla_{0, e_i^0}$  in the fourth term of  $\mathcal{O}'_2$  in (5.26) is 1, thus the contribution of the fourth term of  $\mathcal{O}'_2$  in (5.26) for  $-((\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{O}'_2 P^N)(0, 0)$  is zero. By (5.26), (5.174), (5.176)-(5.179) and (5.181)-(5.184), comparing with (5.117), we get

$$(5.185) \quad -\left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{O}'_2 P^N \right) (0, 0) = \left\{ \frac{1}{2\pi} \left\langle R^{TX_G} \left( \frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_i^0} \right) \frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle \right. \\ + \frac{7}{6} \left[ \frac{5\sqrt{-1}}{2^5 \pi} \left\langle J e_k^\perp, \nabla_{e_k^\perp}^{TY} \left( T \left( \frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0} \right) \right) + \nabla_{\frac{\partial}{\partial z_j^0}}^{TY} \left( T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) \right) \right\rangle + \frac{3}{32\pi} |T(\frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0})|^2 \right. \\ + \frac{3}{16\pi} \left\langle R^{TB} \left( e_k^\perp, \frac{\partial}{\partial z_j^0} \right) e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle + \frac{7}{26\pi} |T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})|^2 - \frac{\sqrt{-1}}{16\pi} \left\langle T(e_k^\perp, J e_k^\perp), T(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0}) \right\rangle \Big] \\ - \frac{1}{8\pi} \left\langle R^{TB} \left( e_k^\perp, \frac{\partial}{\partial z_j^0} \right) e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle + \frac{3}{16\pi} |T(\frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0})|^2 + \frac{3}{32\pi} |T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})|^2 \\ \left. + \frac{3}{64\pi} \left| \sum_j \mathcal{T}_{jj}(\frac{\partial}{\partial \bar{z}_i^0}) \right|^2 + \frac{3}{28\pi} \tilde{\mathcal{T}}_{ijk} (\tilde{\mathcal{T}}_{ijk} + \tilde{\mathcal{T}}_{kji}) + \frac{1}{2\pi} R^{EG} \left( \frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0} \right) \right\} P^N(0, 0).$$

By (5.62) and (5.165),

$$(5.186) \quad -4\pi^2 \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{O}_2'' P^N \right) (0, 0) = -4\pi^2 \left\{ (\mathcal{L}_2^0)^{-1} P^{N^\perp} \right. \\ \left[ -\frac{1}{3} \langle (\nabla^{\cdot TY} \dot{g}^{\cdot TY})_{(\mathcal{R}^0, \mathcal{R}^0)} J \mathcal{R}^\perp + (\nabla^{\cdot TY} \dot{g}^{\cdot TY})_{(\mathcal{R}^\perp, \mathcal{R}^\perp)} J \mathcal{R}^\perp, J \mathcal{R}^\perp \rangle \right. \\ + \frac{1}{6} \langle \nabla_{\mathcal{R}^0}^{TY} (T(e_j^\perp, J_{x_0} e_i^0)) Z_j^\perp Z_i^0 + \nabla_{\mathcal{R}^\perp}^{TY} (T(e_j^0, J_{x_0} e_i^0)) Z_j^0 Z_i^0, J \mathcal{R}^\perp \rangle \\ + \frac{1}{3} \langle R^{TB}(\mathcal{R}^\perp, \mathcal{R}^0) \mathcal{R}^0, \mathcal{R}^\perp \rangle - \frac{1}{12} \sum_l \langle T(\mathcal{R}^0, e_l), J \mathcal{R}^\perp \rangle^2 \\ \left. \left. - \frac{1}{12} \sum_l \langle T(\mathcal{R}^\perp, e_l), J \mathcal{R}^\perp \rangle^2 + \frac{7}{12} |T(\mathcal{R}^\perp, J \mathcal{R}^\perp)|^2 \right] P^N \right\} (0, 0).$$

Now  $\{e_l\} = \{e_i^0\} \cup \{e_k^\perp\}$ .

By (5.107a), (5.119), (5.123), (5.169), (5.180), (5.182), (5.183), (5.186) and comparing with (5.120),

$$\begin{aligned}
 (5.187) \quad & -4\pi^2 \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} \mathcal{O}_2'' P^N \right) (0, 0) = \left\{ \frac{7}{24\pi} \left[ -\frac{8}{3} \nabla_{\frac{\partial}{\partial z_j^0}} \nabla_{\frac{\partial}{\partial \bar{z}_j^0}} \log h \right. \right. \\
 & + \frac{\sqrt{-1}}{3} \left\langle -\nabla_{\frac{\partial}{\partial z_j^0}}^{TY} (T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})) - \nabla_{e_k^\perp}^{TY} (T(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0})), J e_k^\perp \right\rangle \\
 & - \frac{1}{3} \left| T(\frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0}) \right|^2 - \frac{1}{6} \left| T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) \right|^2 - \frac{2}{3} \left\langle R^{TB}(e_k^\perp, \frac{\partial}{\partial z_j^0}) e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle \Bigg] \\
 & - \frac{1}{26\pi} \left\langle (\nabla_{\cdot}^{TY} \dot{g}^{TY})_{(e_j^\perp, e_j^\perp)} J e_k^\perp, J e_k^\perp \right\rangle - \frac{1}{25\pi} \left\langle (\nabla_{\cdot}^{TY} \dot{g}^{TY})_{(e_j^\perp, e_k^\perp)} J e_j^\perp, J e_k^\perp \right\rangle \\
 & - \frac{1}{28\pi} \left( 8 \left| T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) \right|^2 + 4 \left| \sum_j \mathcal{T}_{jj}(\frac{\partial}{\partial \bar{z}_i^0}) \right|^2 + \tilde{\mathcal{T}}_{ijk}(\tilde{\mathcal{T}}_{ijk} + \tilde{\mathcal{T}}_{kji}) \right) \\
 & \left. + \frac{7}{28\pi} (2\mathcal{T}_{jkm}^2 + \mathcal{T}_{jjm} \mathcal{T}_{kkm}) \right\} P^N(0, 0).
 \end{aligned}$$

By (5.73), (5.76), (5.163), (5.185) and (5.187), comparing with (5.100), we have

$$\begin{aligned}
 (5.188) \quad & \Psi_{1,2}(0) = - \left( (\mathcal{L}_2^0)^{-1} P^{N^\perp} (\mathcal{O}_2' + 4\pi^2 \mathcal{O}_2'') P^N \right) (0, 0) \\
 & - \frac{\sqrt{-1}}{16\pi} \left( \left\langle T(e_j^\perp, J e_j^\perp), \tilde{\mu}^E \right\rangle - 2 \left\langle J e_j^\perp, \nabla_{e_j^\perp}^{TY} \tilde{\mu}^E \right\rangle \right) P^N(0, 0) \\
 & = \left\{ \frac{1}{2\pi} \left\langle R^{TX_G}(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_i^0}) \frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle + \frac{1}{2\pi} R^{EG}(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0}) \right. \\
 & + \frac{7}{6} \left[ \frac{1}{6\pi} \Delta_{X_G} \log h + \frac{1}{48\pi} \left\langle R^{TB}(e_k^\perp, \frac{\partial}{\partial z_j^0}) e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle + \frac{1}{96\pi} \left| T(\frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0}) \right|^2 \right. \\
 & - \frac{\sqrt{-1}}{16\pi} \left\langle T(e_k^\perp, J e_k^\perp), T(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0}) \right\rangle + \frac{13}{192\pi} \left| T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) \right|^2 \\
 & \left. + \frac{7\sqrt{-1}}{96\pi} \left\langle \nabla_{\frac{\partial}{\partial z_j^0}}^{TY} (T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})) + \nabla_{e_k^\perp}^{TY} (T(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0})), J e_k^\perp \right\rangle \right] \\
 & - \frac{1}{8\pi} \left\langle R^{TB}(e_k^\perp, \frac{\partial}{\partial z_j^0}) e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0} \right\rangle + \frac{3}{16\pi} \left| T(\frac{\partial}{\partial z_i^0}, \frac{\partial}{\partial \bar{z}_j^0}) \right|^2 + \frac{1}{16\pi} \left| T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) \right|^2 \\
 & + \frac{1}{32\pi} \left| \sum_j \mathcal{T}_{jj}(\frac{\partial}{\partial \bar{z}_i^0}) \right|^2 + \frac{1}{27\pi} \tilde{\mathcal{T}}_{ijk}(\tilde{\mathcal{T}}_{ijk} + \tilde{\mathcal{T}}_{kji}) + \frac{7}{28\pi} (2\mathcal{T}_{jkm}^2 + \mathcal{T}_{jjm} \mathcal{T}_{kkm}) \\
 & - \frac{1}{26\pi} \left\langle (\nabla_{\cdot}^{TY} \dot{g}^{TY})_{(e_j^\perp, e_j^\perp)} J e_k^\perp, J e_k^\perp \right\rangle - \frac{1}{25\pi} \left\langle (\nabla_{\cdot}^{TY} \dot{g}^{TY})_{(e_j^\perp, e_k^\perp)} J e_j^\perp, J e_k^\perp \right\rangle \\
 & \left. - \frac{\sqrt{-1}}{16\pi} \left( \left\langle T(e_j^\perp, J e_j^\perp), \tilde{\mu}^E \right\rangle - 2 \left\langle J e_j^\perp, \nabla_{e_j^\perp}^{TY} \tilde{\mu}^E \right\rangle \right) \right\} P^N(0, 0).
 \end{aligned}$$

By (5.123), (5.130), the term  $\frac{7}{6}[\dots]$  in (5.188) is  $\frac{7}{6} \left( \frac{3}{8\pi} \Delta_{X_G} \log h + \frac{1}{8\pi} \left| T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) \right|^2 \right)$ .

By (5.129) and (5.188), we get (5.162).

The proof of Lemma 5.14 is complete.  $\square$

**Lemma 5.15.** *The following identity holds,*

$$\begin{aligned}
 (5.189) \quad & \left\langle (\nabla_{e_k^\perp}^{TY} \dot{g}_{e_k^\perp}^{TY}) J e_l^\perp, J e_l^\perp \right\rangle = 4 \nabla_{e_k^\perp} \nabla_{e_k^\perp} \log h, \\
 & \left\langle (\nabla_{e_k^\perp}^{TY} \dot{g}_{e_l^\perp}^{TY}) J e_l^\perp, J e_k^\perp \right\rangle = 4 \nabla_{e_k^\perp} \nabla_{e_k^\perp} \log h + 2 \left| \sum_l \mathcal{T}_l \left( \frac{\partial}{\partial \bar{z}_j^0} \right) \right|^2 \\
 & \quad - 2 |T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})|^2 + \frac{1}{2} (\tilde{\mathcal{T}}_{jki} + \tilde{\mathcal{T}}_{ijk}) \tilde{\mathcal{T}}_{ijk}.
 \end{aligned}$$

*Proof.* By using the same argument as in (5.119), we get the first equation of (5.189).

Recall that  $P^{T^H X}, P^{TY}$  are the projections from  $TX = T^H X \oplus TY$  onto  $T^H X, TY$ .

By (1.3), (1.7), (3.1), (3.40) and (3.41) (cf. also (5.31)),

$$(5.190a) \quad (P^{T^H X} J e_l^{\perp, H})|_{\mu^{-1}(0)} = 0, \quad (J e_l^{\perp, H})_{x_0} \in TY,$$

$$(5.190b) \quad (\nabla_{e_k^{\perp, H}}^{TX} e_l^{\perp, H})_{x_0} = -\frac{1}{2} T(e_k^\perp, e_l^\perp),$$

$$(\nabla_{e_j^0}^{TX} e_l^{\perp, H})_{x_0} = (A(e_j^0) e_l^\perp)^H - \frac{1}{2} T(e_j^0, e_l^\perp),$$

$$(5.190c) \quad (\nabla_{J e_l^{\perp, H}}^{TX} e_k^H)_{x_0} = \frac{1}{2} \langle T(e_k, e_j), J e_l^\perp \rangle e_j^H + \langle T(e_k, J e_l^\perp), J e_j^\perp \rangle J e_j^\perp.$$

From (5.190a), we get

$$(5.191) \quad \nabla_{e_i^0}^{TX} P^{T^H X} J e_l^{\perp, H} = \nabla_{J e_k^{\perp, H}}^{TX} P^{T^H X} J e_l^{\perp, H} = 0.$$

By (3.40), (5.14), (5.71) and (5.190b), we get at  $x_0$ ,

$$\begin{aligned}
 (5.192) \quad & \nabla_{e_k^{\perp, H}}^{TX} P^{T^H X} J e_l^{\perp, H} = \nabla_{e_k^{\perp, H}}^{T^H X} P^{T^H X} J e_l^{\perp, H} \\
 & = -\frac{1}{2} J T(e_k^\perp, e_l^\perp) + \frac{1}{2} \langle J T(e_k^\perp, e_j), e_l^\perp \rangle e_j \\
 & = -\frac{1}{2} (\tilde{\mathcal{T}}_{klj} - \tilde{\mathcal{T}}_{kjl}) e_j^\perp + \frac{1}{2} \langle J T(e_k^\perp, e_j^0), e_l^\perp \rangle e_j^0.
 \end{aligned}$$

By (5.6a), (5.190b) and (5.190c), at  $x_0$ ,

$$\begin{aligned}
 (5.193) \quad & [e_k^{\perp, H}, J e_l^{\perp, H}] = \nabla_{e_k^{\perp, H}}^{TX} J e_l^{\perp, H} - \nabla_{J e_l^{\perp, H}}^{TX} e_k^{\perp, H} = J \nabla_{e_k^{\perp, H}}^{TX} e_l^{\perp, H} - \nabla_{J e_l^{\perp, H}}^{TX} e_k^{\perp, H} \\
 & = -\frac{1}{2} J T(e_k^\perp, e_l^\perp) + \frac{1}{2} \langle J T(e_k^\perp, e_j), e_l^\perp \rangle e_j - \langle T(e_k^\perp, J e_j^\perp), J e_l^\perp \rangle J e_j^\perp.
 \end{aligned}$$

By (5.14), (5.191), (5.192) and (5.193), we get at  $x_0$ ,

$$\begin{aligned}
 (5.194) \quad & \left\langle \nabla_{[e_k^{\perp, H}, J e_l^{\perp, H}]}^{TX} P^{T^H X} J e_l^{\perp, H}, e_k^{\perp, H} \right\rangle \\
 & = -\frac{1}{2} \left\langle \nabla_{J T(e_k^\perp, e_l^\perp) - \langle J T(e_k^\perp, e_j^\perp), e_l^\perp \rangle e_j^\perp}^{TX} P^{T^H X} J e_l^{\perp, H}, e_k^{\perp, H} \right\rangle \\
 & = \frac{1}{4} (\tilde{\mathcal{T}}_{klj} - \tilde{\mathcal{T}}_{kjl}) (\tilde{\mathcal{T}}_{jlk} - \tilde{\mathcal{T}}_{jkl}).
 \end{aligned}$$

By (5.6a), (5.190b), (5.191) and (5.192), at  $x_0$ ,

$$\begin{aligned}
 \nabla_{e_j^0, H}^{TX} P^{TY} J e_l^{\perp, H} &= J \nabla_{e_j^0, H}^{TX} e_l^{\perp, H} = J A(e_j^0) e_l^{\perp} - \frac{1}{2} J T(e_j^0, e_l^{\perp}), \\
 (5.195) \quad \nabla_{e_k^{\perp, H}}^{TX} P^{TY} J e_l^{\perp, H} &= J \nabla_{e_k^{\perp, H}}^{TX} e_l^{\perp, H} - \nabla_{e_k^{\perp, H}}^{TX} P^{THX} J e_l^{\perp, H} \\
 &= -\frac{1}{2} \langle J T(e_k^{\perp}, e_j), e_l^{\perp} \rangle e_j = -\frac{1}{2} \tilde{T}_{kjl} e_j^{\perp} - \frac{1}{2} \langle J T(e_k^{\perp}, e_j^0), e_l^{\perp} \rangle e_j^0.
 \end{aligned}$$

Thus by (5.195), at  $x_0$ ,

$$(5.196) \quad \nabla_{e_k^{\perp, H}}^{TY} P^{TY} J e_l^{\perp, H} = P^{TY} \nabla_{e_k^{\perp, H}}^{TX} P^{TY} J e_l^{\perp, H} = 0.$$

By (1.3), (1.6), (1.7) and (5.196), at  $x_0$ ,

$$\begin{aligned}
 (5.197) \quad \langle (\nabla_{e_k^{\perp}}^{TY} \dot{g}_{e_l^{\perp}}^{TY}) J e_l^{\perp}, J e_k^{\perp} \rangle &= e_k^{\perp} \langle \dot{g}_{e_l^{\perp}}^{TY} P^{TY} J e_l^{\perp}, P^{TY} J e_k^{\perp} \rangle = 2e_k^{\perp} \langle \nabla_{P^{TY} J e_l^{\perp}}^{TX} e_l^{\perp}, P^{TY} J e_k^{\perp} \rangle \\
 &= 2e_k^{\perp} \langle \nabla_{J e_l^{\perp}}^{TX} e_l^{\perp}, J e_k^{\perp} \rangle - 2e_k^{\perp} \langle \nabla_{P^{THX} J e_l^{\perp}}^{TX} e_l^{\perp}, J e_k^{\perp} \rangle - 2e_k^{\perp} \langle \nabla_{P^{TY} J e_l^{\perp}}^{TX} e_l^{\perp}, P^{THX} J e_k^{\perp} \rangle.
 \end{aligned}$$

By (5.5d), (5.14), (5.190a), (5.190b) and (5.192), at  $x_0$

$$\begin{aligned}
 (5.198) \quad -2e_k^{\perp} \langle \nabla_{P^{THX} J e_l^{\perp}}^{TX} e_l^{\perp}, J e_k^{\perp} \rangle &= -2 \langle \nabla_{\nabla_{e_k^{\perp}}^{TX} P^{THX} J e_l^{\perp}}^{TX} e_l^{\perp}, J e_k^{\perp} \rangle \\
 &= \left\langle T \left( -\frac{1}{2} (\tilde{T}_{klj} - \tilde{T}_{kjl}) e_j^{\perp} + \frac{1}{2} \langle J T(e_k^{\perp}, e_j^0), e_l^{\perp} \rangle e_j^0, e_l^{\perp} \right), J e_k^{\perp} \right\rangle \\
 &= \frac{1}{2} (\tilde{T}_{klj} - \tilde{T}_{kjl}) \tilde{T}_{jlk} + \frac{1}{2} \langle T(e_j^0, e_l^{\perp}), J e_k^{\perp} \rangle \langle J T(e_k^{\perp}, e_j^0), e_l^{\perp} \rangle \\
 &= \frac{1}{2} (\tilde{T}_{klj} - \tilde{T}_{kjl}) \tilde{T}_{jlk} + \frac{1}{2} |T(e_k^{\perp}, e_j^0)|^2.
 \end{aligned}$$

By (5.5e), (5.14), (5.190a), (5.190b) and (5.192), at  $x_0$ , we have

$$\begin{aligned}
 (5.199) \quad -2e_k^{\perp} \langle \nabla_{P^{TY} J e_l^{\perp}}^{TX} e_l^{\perp}, P^{THX} J e_k^{\perp} \rangle &= -2 \langle \nabla_{P^{TY} J e_l^{\perp}}^{TX} e_l^{\perp}, \nabla_{e_k^{\perp}}^{TX} P^{THX} J e_k^{\perp} \rangle \\
 &= -\frac{1}{2} \langle J T(e_k^{\perp}, e_j), e_k^{\perp} \rangle \langle T(e_l^{\perp}, e_j), J e_l^{\perp} \rangle = \frac{1}{2} \mathcal{T}_{ll}(e_j^0) \mathcal{T}_{kk}(e_j^0).
 \end{aligned}$$

Now by (5.6a),

$$\begin{aligned}
 (5.200) \quad e_k^{\perp} \langle \nabla_{J e_l^{\perp}}^{TX} e_l^{\perp}, J e_k^{\perp} \rangle &= -e_k^{\perp} \langle \nabla_{J e_l^{\perp}}^{TX} J e_l^{\perp}, e_k^{\perp} \rangle \\
 &= -e_k^{\perp} \langle \nabla_{P^{TY} J e_l^{\perp}}^{TX} P^{TY} J e_l^{\perp} + \nabla_{P^{THX} J e_l^{\perp}}^{TX} P^{TY} J e_l^{\perp} + \nabla_{J e_l^{\perp}}^{TX} P^{THX} J e_l^{\perp}, e_k^{\perp} \rangle.
 \end{aligned}$$

By Theorem 5.1, (1.7), (5.190a) and (5.192), at  $x_0$ ,

$$\begin{aligned}
 (5.201) \quad & -2e_k^\perp \left\langle \nabla_{P^{TH}X J e_l^\perp}^{TX} P^{TY} J e_l^\perp, e_k^\perp \right\rangle = -2 \left\langle \nabla_{\nabla_{e_k^\perp}^{TX} P^{TH}X J e_l^\perp}^{TX} P^{TY} J e_l^\perp, e_k^\perp \right\rangle \\
 & = - \left\langle T \left( -\frac{1}{2}(\tilde{\mathcal{T}}_{klj} - \tilde{\mathcal{T}}_{kjl})e_j^\perp + \frac{1}{2} \langle JT(e_k^\perp, e_j^0), e_l^\perp \rangle e_j^\perp, e_k^\perp \right), J e_l^\perp \right\rangle \\
 & = -\frac{1}{2}(\tilde{\mathcal{T}}_{klj} - \tilde{\mathcal{T}}_{kjl})\tilde{\mathcal{T}}_{jkl} - \frac{1}{2}|T(e_k^\perp, e_j^0)|^2.
 \end{aligned}$$

And by (5.5d), (5.190a)-(5.190c), (5.191), (5.192), (5.194) and (5.201), at  $x_0$ ,

$$\begin{aligned}
 (5.202) \quad & -2e_k^\perp \left\langle \nabla_{J e_l^\perp}^{TX} P^{TH}X J e_l^\perp, e_k^\perp \right\rangle = -2 \left\langle \left( \nabla_{J e_l^\perp}^{TX} \nabla_{e_k^\perp}^{TX} + \nabla_{[e_k^\perp, H, J e_l^\perp, H]}^{TX} \right) P^{TH}X J e_l^\perp, e_k^\perp \right\rangle \\
 & = - \left\langle T \left( -\frac{1}{2}(\tilde{\mathcal{T}}_{klj} - \tilde{\mathcal{T}}_{kjl})e_j^\perp + \frac{1}{2} \langle JT(e_k^\perp, e_j), e_l^\perp \rangle e_j, e_k^\perp \right), J e_l^\perp \right\rangle - \frac{1}{2}(\tilde{\mathcal{T}}_{klj} - \tilde{\mathcal{T}}_{kjl})(\tilde{\mathcal{T}}_{jlk} - \tilde{\mathcal{T}}_{jkl}) \\
 & = -\frac{1}{2}(\tilde{\mathcal{T}}_{klj} - \tilde{\mathcal{T}}_{kjl})\tilde{\mathcal{T}}_{jlk} - \frac{1}{2}|T(e_k^\perp, e_j^0)|^2.
 \end{aligned}$$

Finally, by (1.4), (1.7), (1.24) and (5.196), as in (5.119),

$$\begin{aligned}
 (5.203) \quad & -2e_k^\perp \left\langle \nabla_{P^{TY} J e_l^\perp}^{TX} P^{TY} J e_l^\perp, e_k^\perp \right\rangle = 2e_k^\perp \langle T(e_k^\perp, P^{TY} J e_l^\perp), P^{TY} J e_l^\perp \rangle \\
 & = \left\langle (\nabla_{e_k^\perp}^{TY} \dot{g}_{e_k^\perp}^{TY}) J e_l^\perp, J e_l^\perp \right\rangle = 4 \nabla_{e_k^\perp} \nabla_{e_k^\perp} \log h.
 \end{aligned}$$

Thus by (5.197)-(5.203),

$$\begin{aligned}
 (5.204) \quad & \left\langle (\nabla_{e_k^\perp}^{TY} \dot{g}_{e_k^\perp}^{TY}) J e_l^\perp, J e_k^\perp \right\rangle = 4 \nabla_{e_k^\perp} \nabla_{e_k^\perp} \log h - \frac{1}{2}|T(e_k^\perp, e_j^0)|^2 \\
 & \quad + \frac{1}{2}\mathcal{T}_{ll}(e_j^0)\mathcal{T}_{kk}(e_j^0) - \frac{1}{2}(\tilde{\mathcal{T}}_{klj} - \tilde{\mathcal{T}}_{kjl})\tilde{\mathcal{T}}_{jkl}.
 \end{aligned}$$

From (3.6) and (5.204), we get (5.189).  $\square$

*Proof of Theorem 0.7.* By (5.14), (5.94),

$$\begin{aligned}
 (5.205) \quad & \sum_k \mathcal{F}_1(e_k^\perp)^2 = -\langle \tilde{\mu}_{x_0}^E, \tilde{\mu}_{x_0}^E \rangle_{g^{TY}} - \left\langle \tilde{\mu}^E, \frac{3}{2}\sqrt{-1}T(e_l^\perp, J e_l^\perp) + 2T\left(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0}\right) \right\rangle \\
 & \quad + \left| \sum_j T\left(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0}\right) \right|^2 + \frac{9}{16}\mathcal{T}_{llm}\mathcal{T}_{kkm} - \frac{3\sqrt{-1}}{2} \left\langle T(e_l^\perp, J e_l^\perp), T\left(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0}\right) \right\rangle, \\
 & \quad \mathcal{F}_1(e_k^\perp)\mathcal{T}_{kll} = -\sqrt{-1} \left\langle T(e_l^\perp, J e_l^\perp), \tilde{\mu}^E + T\left(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0}\right) \right\rangle + \frac{3}{4}\mathcal{T}_{llm}\mathcal{T}_{kkm}.
 \end{aligned}$$

By (5.137), (5.138), (5.139) and (5.205), we have

$$\begin{aligned}
(5.206) \quad & (\Psi_{1,1} + \Psi_{1,1}^* + \Psi_{1,3} - \Psi_{1,4})(0) = \left\{ -\frac{1}{2\pi} \sum_k \mathcal{F}_1(e_k^\perp)^2 - \frac{1}{8\pi} \mathcal{F}_1(e_k^\perp) \mathcal{T}_{kl} \right. \\
& - \frac{13}{2^6 \cdot 3\pi} \mathcal{T}_{klm}^2 + \frac{1}{2^7\pi} \mathcal{T}_{kkm} \mathcal{T}_{llm} - \frac{11}{48\pi} \left| \mathcal{T}_{kl} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) \right|^2 - \frac{1}{8\pi} \left| \sum_k \mathcal{T}_{kk} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) \right|^2 \Big\} P^N(0, 0) \\
& = \left\{ \frac{1}{2\pi} \langle \tilde{\mu}_{x_0}^E, \tilde{\mu}_{x_0}^E \rangle_{g^{TY}} + \frac{1}{\pi} \left\langle \tilde{\mu}^E, \frac{7}{8} \sqrt{-1} T(e_l^\perp, J e_l^\perp) + T\left(\frac{\partial}{\partial \bar{z}_l^0}, \frac{\partial}{\partial \bar{z}_l^0}\right) \right\rangle \right. \\
& - \frac{1}{2\pi} \left| \sum_j T\left(\frac{\partial}{\partial \bar{z}_j^0}, \frac{\partial}{\partial \bar{z}_j^0}\right) \right|^2 + \frac{7\sqrt{-1}}{8\pi} \left\langle T(e_l^\perp, J e_l^\perp), T\left(\frac{\partial}{\partial \bar{z}_j^0}, \frac{\partial}{\partial \bar{z}_j^0}\right) \right\rangle - \frac{47}{2^7\pi} \mathcal{T}_{kkm} \mathcal{T}_{llm} \\
& \left. - \frac{13}{2^6 \cdot 3\pi} \mathcal{T}_{klm}^2 - \frac{11}{48\pi} \left| T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0}) \right|^2 - \frac{1}{8\pi} \left| \sum_k \mathcal{T}_{kk} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) \right|^2 \right\} P^N(0, 0).
\end{aligned}$$

By (5.162) and (5.189), we get

$$\begin{aligned}
(5.207) \quad & \Psi_{1,2}(0) + \Psi_{1,2}(0)^* = \left\{ \frac{1}{8\pi} r_{x_0}^{X_G} + \frac{1}{\pi} R^{EG} \left( \frac{\partial}{\partial \bar{z}_j^0}, \frac{\partial}{\partial \bar{z}_j^0} \right) + \frac{1}{\pi} \Delta_{X_G} \log h \right. \\
& - \frac{3}{8\pi} \nabla_{e_k^\perp} \nabla_{e_k^\perp} \log h + \frac{35}{48\pi} |T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})|^2 + \frac{\sqrt{-1}}{8\pi} \left\langle T(e_l^\perp, J e_l^\perp), T\left(\frac{\partial}{\partial \bar{z}_j^0}, \frac{\partial}{\partial \bar{z}_j^0}\right) \right\rangle \\
& + \frac{1}{2\pi} \left| T\left(\frac{\partial}{\partial \bar{z}_i^0}, \frac{\partial}{\partial \bar{z}_j^0}\right) \right|^2 - \frac{1}{16\pi} \left| \sum_k \mathcal{T}_{kk} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) \right|^2 + \frac{1}{2^6\pi} \left[ \tilde{\mathcal{T}}_{ijk} (\tilde{\mathcal{T}}_{kji} + \tilde{\mathcal{T}}_{ijk}) - 2(\tilde{\mathcal{T}}_{jki} + \tilde{\mathcal{T}}_{ijk}) \tilde{\mathcal{T}}_{ijk} \right] \\
& \left. + \frac{7}{2^7\pi} (2\mathcal{T}_{jkm}^2 + \mathcal{T}_{jjm} \mathcal{T}_{kkm}) - \frac{\sqrt{-1}}{8\pi} \left( \langle T(e_l^\perp, J e_l^\perp), \tilde{\mu}^E \rangle - 2 \langle J e_k^\perp, \nabla_{e_k^\perp}^{TY} \tilde{\mu}^E \rangle \right) \right\} P^N(0, 0).
\end{aligned}$$

Thus by (5.132), (5.206) and (5.207), as  $\tilde{\mathcal{T}}_{ijk}$  is anti-symmetric on  $i, j$ , we get

$$\begin{aligned}
(5.208) \quad & P^{(2)}(0, 0) = \left\{ \frac{1}{8\pi} r_{x_0}^{X_G} + \frac{1}{\pi} R^{EG} \left( \frac{\partial}{\partial \bar{z}_j^0}, \frac{\partial}{\partial \bar{z}_j^0} \right) + \frac{1}{\pi} \Delta_{X_G} \log h - \frac{3}{8\pi} \nabla_{e_k^\perp} \nabla_{e_k^\perp} \log h \right. \\
& + \frac{1}{2\pi} |T(e_k^\perp, \frac{\partial}{\partial \bar{z}_j^0})|^2 + \frac{1}{2\pi} \left| T\left(\frac{\partial}{\partial \bar{z}_i^0}, \frac{\partial}{\partial \bar{z}_j^0}\right) \right|^2 - \frac{1}{2\pi} \left| \sum_j T\left(\frac{\partial}{\partial \bar{z}_j^0}, \frac{\partial}{\partial \bar{z}_j^0}\right) \right|^2 \\
& + \frac{\sqrt{-1}}{\pi} \left\langle T(e_l^\perp, J e_l^\perp), T\left(\frac{\partial}{\partial \bar{z}_j^0}, \frac{\partial}{\partial \bar{z}_j^0}\right) \right\rangle - \frac{3}{16\pi} \left| \sum_k \mathcal{T}_{kk} \left( \frac{\partial}{\partial \bar{z}_i^0} \right) \right|^2 + \frac{1}{24\pi} \mathcal{T}_{klm}^2 \\
& - \frac{5}{16\pi} \mathcal{T}_{kkm} \mathcal{T}_{llm} + \frac{1}{2^6\pi} \tilde{\mathcal{T}}_{ijk} (3\tilde{\mathcal{T}}_{kji} - \tilde{\mathcal{T}}_{ijk}) + \frac{1}{2\pi} \langle \tilde{\mu}_{x_0}^E, \tilde{\mu}_{x_0}^E \rangle_{g^{TY}} \\
& \left. + \frac{1}{\pi} \left\langle \tilde{\mu}^E, \frac{3}{4} \sqrt{-1} T(e_l^\perp, J e_l^\perp) + T\left(\frac{\partial}{\partial \bar{z}_l^0}, \frac{\partial}{\partial \bar{z}_l^0}\right) \right\rangle + \frac{\sqrt{-1}}{4\pi} \left\langle J e_k^\perp, \nabla_{e_k^\perp}^{TY} \tilde{\mu}^E \right\rangle \right\} P^N(0, 0).
\end{aligned}$$

By Theorem 5.1, (1.4), (1.24), (5.5c) and (5.14), as same as in (5.119), we get for  $U \in T_{x_0} X_G$ ,

$$\begin{aligned}
(5.209) \quad & \mathcal{T}_{llm} = \langle T(e_m^\perp, J e_l^\perp), J e_l^\perp \rangle = 2 \nabla_{e_m^\perp} \log h, \quad T(e_l^\perp, J e_l^\perp) = 2(\nabla_{e_k^\perp} \log h) J e_k^\perp, \\
& \mathcal{T}_{kk}(U) = -2 \langle T(JU, J e_k^\perp), J e_k^\perp \rangle = -\langle \dot{g}_{JU}^{TY} J e_k^\perp, J e_k^\perp \rangle = -4 \nabla_{JU} \log h.
\end{aligned}$$



By (5.136), (5.208) and (5.209), we get Theorem 0.7.  $\square$

**5.6. Coefficient  $\Phi_1$ : general case.** We use the general assumption at the beginning of this Section, but we do not suppose that  $\mathbf{J} = J$  in (0.2).

Let  $\bar{\partial}^{L^p \otimes E, *}$  be the formal adjoint of the Dolbeault operator  $\bar{\partial}^{L^p \otimes E}$  on the Dolbeault complex  $\Omega^{0, \bullet}(X, L^p \otimes E)$  with the scalar product  $\langle \cdot \rangle$  induced by  $g^{TX}$ ,  $h^L$ ,  $h^E$  as in Section 2.2.

Set

$$(5.210) \quad D_p = \sqrt{2} \left( \bar{\partial}^{L^p \otimes E} + \bar{\partial}^{L^p \otimes E, *} \right)$$

Then

$$(5.211) \quad D_p^2 = 2 \left( \bar{\partial}^{L^p \otimes E} \bar{\partial}^{L^p \otimes E, *} + \bar{\partial}^{L^p \otimes E, *} \bar{\partial}^{L^p \otimes E} \right)$$

preserves the  $\mathbb{Z}$ -grading of  $\Omega^{0, \bullet}(X, L^p \otimes E)$ .

For  $p$  large enough,

$$(5.212) \quad \text{Ker } D_p = \text{Ker } D_p^2 = H^0(X, L^p \otimes E).$$

Here  $D_p$  need not be a  $\text{spin}^c$  Dirac operator on  $\Omega^{0, \bullet}(X, L^p \otimes E)$ .

Let  $P_p^G(x, x')$  ( $x, x' \in X$ ) be the smooth kernel of the orthogonal projection  $P_p^G$  from  $(\mathcal{C}^\infty(X, L^p \otimes E), \langle \cdot \rangle)$  onto  $(\text{Ker } D_p^2)^G$  with respect to the Riemannian volume form  $dv_X(x')$  for  $p$  large enough.

We explain now how to reduce the study of the asymptotic expansion of  $P_p^G(x, x')$  to the  $\mathbf{J} = J$  case.

Let  $g_\omega^{TX}(\cdot, \cdot) := \omega(\cdot, J\cdot)$  be the metric on  $TX$  induced by  $\omega, J$ . We will use a subscript  $\omega$  to indicate the objects corresponding to  $g_\omega^{TX}$ , especially  $r_\omega^X$  is the scalar curvature of  $(TX, g_\omega^{TX})$ , and  $\Delta_{X_G, \omega}$  is the Bochner-Laplace operator on  $X_G$  as in (1.21) associated to  $g_\omega^{TX_G}$ .

Let  $\det_{\mathbb{C}}$  denote the determinant function on the complex bundle  $T^{(1,0)}X$ , and  $|\mathbf{J}| = (-\mathbf{J}^2)^{-1/2}$ .

Let  $h_\omega^E := (\det_{\mathbb{C}}|\mathbf{J}|)^{-1} h^E$  define a metric on  $E$ . Let  $R_\omega^E$  be the curvature associated to the holomorphic Hermitian connection on  $(E, h_\omega^E)$ .

Let  $\langle \cdot \rangle_\omega$  be the Hermitian product on  $\mathcal{C}^\infty(X, L^p \otimes E)$  induced by  $g_\omega^{TX}, h^L, h_\omega^E$  as in (1.19), then

$$(5.213) \quad (\mathcal{C}^\infty(X, L^p \otimes E), \langle \cdot \rangle_\omega) = (\mathcal{C}^\infty(X, L^p \otimes E), \langle \cdot \rangle), \quad dv_{X, \omega} = (\det_{\mathbb{C}}|\mathbf{J}|) dv_X.$$

Observe that  $H^0(X, L^p \otimes E)$  does not depend on  $g^{TX}, h^L, h^E$ .

Let  $P_{\omega, p}^G(x, x')$  ( $x, x' \in X$ ) be the smooth kernel of the orthogonal projection  $P_{\omega, p}^G$  from  $(\mathcal{C}^\infty(X, L^p \otimes E), \langle \cdot \rangle_\omega)$  onto  $H^0(X, L^p \otimes E)^G$  with respect to  $dv_{X, \omega}(x)$ .

By (5.213),

$$(5.214) \quad P_p^G(x, x') = (\det_{\mathbb{C}}|\mathbf{J}|)(x') P_{\omega, p}^G(x, x').$$

We will use the trivialization in Introduction corresponding to  $g_\omega^{TX}$ .

Since  $g_\omega^{TX}(\cdot, \cdot) = \omega(\cdot, J\cdot)$  is a Kähler metric on  $TX$ ,  $D_{\omega, p}$  is a Dirac operator (cf. Def. 2.1). Thus Theorems 0.1, 0.2 hold for  $P_{\omega, p}^G(x, x')$ .

Let  $dv_B$  be the volume form on  $B$  induced by  $g^{TX}$  as in Introduction.

As in (0.11), let  $\tilde{\kappa} \in \mathcal{C}^\infty(TB|_{X_G}, \mathbb{R})$  be defined by for  $Z \in T_{x_0}B$ ,  $x_0 \in X_G$ ,

$$(5.215) \quad dv_B(x_0, Z) = \tilde{\kappa}(x_0, Z) dv_{X_G, \omega}(x_0) dv_{N_G, \omega, x_0}.$$

As in (0.17), we introduce  $\mathcal{J}_p(x_0)$  a section of  $\text{End}(E_G)$  on  $X_G$ ,

$$(5.216) \quad \mathcal{J}_p(x_0) = \int_{Z \in N_G, |Z| \leq \varepsilon_0} h^2(x_0, Z) P_p^G \circ \Psi((x_0, Z), (x_0, Z)) \tilde{\kappa}(x_0, Z) dv_{N_G, \omega, x_0}.$$

Then (0.18) still holds.

Summarizes, we have the following result,

**Theorem 5.16.** *The smooth kernel  $P_p^G(x, x')$  has a full off-diagonal asymptotic expansion analogous to (0.14) with  $\mathcal{Q}_0 = (\det_{\mathbb{C}} |\mathbf{J}|) \text{Id}_{E_G}$  as  $p \rightarrow \infty$ . There exist  $\Phi_r(x_0) \in \text{End}(E_G)_{x_0}$  polynomials in  $A_\omega$ ,  $R_\omega^{TB}$ ,  $R^{EB}$ ,  $\mu^E$ ,  $R^E$  (resp.  $h_\omega$ ,  $R^{LB}$ ; resp.  $\mu$ ) and their derivatives at  $x_0$  to order  $2r-1$  (resp.  $2r$ , resp.  $2r+1$ ), and  $\Phi_0 = \text{Id}_{E_G}$  such that (0.25) holds for  $\mathcal{J}_p$ . Moreover*

$$(5.217) \quad \Phi_1(x_0) = \frac{1}{8\pi} \left[ r_\omega^{X_G} + 6\Delta_{X_G, \omega} \log(h_\omega|_{X_G}) - 2\Delta_{X_G, \omega} \left( \log(\det_{\mathbb{C}} |\mathbf{J}|) \right) + 4R^{EG}(w_{\omega, j}^0, \bar{w}_{\omega, j}^0) \right].$$

Here  $\{w_{\omega, j}\}$  is an orthogonal basis of  $(T^{(1,0)}X_G, g_\omega^{TX_G})$ .

*Proof.* By (5.213)-(5.216),

$$(5.218) \quad \mathcal{J}_p(x_0) = \int_{Z \in N_G, |Z| \leq \varepsilon_0} h_\omega^2(x_0, Z) P_{\omega, p}^G \circ \Psi((x_0, Z), (x_0, Z)) \kappa_\omega(x_0, Z) dv_{N_G, \omega}(Z).$$

From the above discussion, only (5.217) reminds to be proved. But

$$(5.219) \quad R_\omega^{EG} = R^{EG} - \bar{\partial}\partial \log(\det_{\mathbb{C}} |\mathbf{J}|),$$

Thus

$$(5.220) \quad 2R_\omega^{EG}(w_{\omega, j}^0, \bar{w}_{\omega, j}^0) = 2R^{EG}(w_{\omega, j}^0, \bar{w}_{\omega, j}^0) - \Delta_{X_G, \omega} \log(\det_{\mathbb{C}} |\mathbf{J}|),$$

and (5.217) is from (0.7) and (5.218).  $\square$

## 6. BERGMAN KERNEL AND GEOMETRIC QUANTIZATION

In this Section, we prove Theorems 0.10, 0.11.

*Proof of Theorem 0.10.* We use the notation in Section 4.4.

By Theorem 4.4 and Lemma 4.6, we know that  $p^{-\frac{n_0}{4}}(\sigma_p \circ \sigma_p^*)^{\frac{1}{2}}$  is a Toeplitz operator with principal symbol  $(2^{\frac{n_0}{4}}/\tilde{h}(x_0)) \text{Id}_{E_G}$  in the sense of Def. 4.3, and its kernel has an expansion analogous to (4.45) and  $Q_{0,0}$  therein is  $2^{\frac{n_0}{4}}/\tilde{h}(x_0)$ .

We claim that

$$(6.1) \quad \mathbb{T}_p = p^{-\frac{n_0}{2}}(\sigma_p \circ \sigma_p^*)^{\frac{1}{2}} \tilde{h}^2(\sigma_p \circ \sigma_p^*)^{\frac{1}{2}}$$

is a Toeplitz operator with principal symbol  $2^{\frac{n_0}{2}} \text{Id}_{E_G}$ .

Indeed, when  $E = \mathbb{C}$ , this is a consequence of [9] on the composition of the Toeplitz operators.

To get the above claim for general  $E$ , we need just keep in mind that the kernel  $\mathbb{T}_p(x_0, x'_0)$  of  $\mathbb{T}_p$  with respect to  $dv_{X_G}(x'_0)$  has the expansion analogous to (4.45) and  $Q_{0,0}$  therein is  $2^{\frac{n_0}{2}} \text{Id}_{E_G}$ .

Our claim then follows from the composition of the expansion of the kernel of  $p^{-\frac{n_0}{4}}(\sigma_p \circ \sigma_p^*)^{\frac{1}{2}}$ , as well as the Taylor expansion of  $\tilde{h}^2$  (cf. also the recent book [28] for a more detailed proof).

Now we still denote by  $\langle \cdot, \cdot \rangle$  the  $L^2$ -scalar product on  $\mathcal{C}^\infty(X_G, L_G^p \otimes E_G)$  induced by  $h^{L_G^p}, h^{E_G}, g^{TX_G}$  as in (1.19).

Let  $\{s_i^p\}$  be an orthonormal basis of  $(H^0(X, L^p \otimes E)^G, \langle \cdot, \cdot \rangle)$ , then  $\varphi_i^p = (\sigma_p \circ \sigma_p^*)^{-\frac{1}{2}} \sigma_p s_i^p$  is an orthonormal basis of  $(H^0(X_G, L_G^p \otimes E_G), \langle \cdot, \cdot \rangle)$ .

From Def. 4.3, (0.28), (1.19) and (6.1), we get

$$(6.2) \quad (2p)^{-\frac{n_0}{2}} \langle \sigma_p s_i^p, \sigma_p s_j^p \rangle_{\tilde{h}} = (2p)^{-\frac{n_0}{2}} \left\langle (\sigma_p \circ \sigma_p^*)^{\frac{1}{2}} \varphi_i^p, (\sigma_p \circ \sigma_p^*)^{\frac{1}{2}} \varphi_j^p \right\rangle_{\tilde{h}} \\ = 2^{-\frac{n_0}{2}} \langle \mathbb{T}_p \varphi_i^p, \varphi_j^p \rangle = \delta_{ij} + \mathcal{O}\left(\frac{1}{p}\right).$$

The proof of Theorem 0.10 is complete.  $\square$

*Proof of Theorem 0.11.* Set

$$\tilde{h}^{E_G} = \tilde{h}^2 h^{E_G}.$$

Then  $\tilde{P}_p^{X_G}$  is the orthogonal projection from  $\mathcal{C}^\infty(X_G, L_G^p \otimes E_G)$  onto  $H^0(X, L_G^p \otimes E_G)$ , associated to the Hermitian product on  $\mathcal{C}^\infty(X_G, L_G^p \otimes E_G)$  induced by the metrics  $h^{L_G^p}, \tilde{h}^{E_G}, g^{TX_G}$  as in (1.19).

Let  $\tilde{P}_{p,\omega}^{X_G}(x_0, x'_0)$  be the smooth kernel of  $\tilde{P}_p^{X_G}$  with respect to  $dv_{X_G}(x'_0)$ . Then

$$(6.3) \quad \tilde{P}_{p,\omega}^{X_G}(x_0, x'_0) = \tilde{h}^2(x'_0) \tilde{P}_p^{X_G}(x_0, x'_0).$$

Let  $\tilde{\nabla}^{E_G}$  be the Hermitian holomorphic connection on  $(E_G, \tilde{h}^{E_G})$  with curvature  $\tilde{R}^{E_G}$ .

Then

$$(6.4) \quad \tilde{\nabla}^{E_G} = \nabla^{E_G} + \partial \log(\tilde{h}^2), \quad \tilde{R}^{E_G} = R^{E_G} + 2\bar{\partial}\partial \log \tilde{h}.$$

Thus from (6.4),

$$(6.5) \quad \tilde{R}^{E_G}(w_j^0, \bar{w}_j^0) = 2\tilde{R}^{E_G}\left(\frac{\partial}{\partial z_j^0}, \frac{\partial}{\partial \bar{z}_j^0}\right) = R^{E_G}(w_j^0, \bar{w}_j^0) + \Delta_{X_G} \log \tilde{h}.$$

By (5.18), (6.3) and (6.5), Theorem 0.11 is a direct consequence of [17, Theorem 1.3] (or Theorem 0.6 with  $G = \{1\}$ ) for  $\tilde{P}_{p,\omega}^{X_G}(x_0, x_0)$ .  $\square$

**Acknowledgments.** We thank Professor Jean-Michel Bismut for useful conversations. The work of the second author was partially supported by MOEC, MOSTC and NNSFC. Part of work was done while the second author was visiting IHES during February and March, 2005. He would like to thank Professor Jean-Pierre Bourguignon for the hospitality.

## REFERENCES

- [1] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Grundle Math. Wiss. Band 298, Springer-Verlag, Berlin, 1992.
- [2] J.-M. Bismut, *The Atiyah-Singer index theorem for families of Dirac operators: two heat equation proofs*, Invent. Math. **83** (1986), no. 1, 91–151.
- [3] ———, *Koszul complexes, harmonic oscillators, and the Todd class*, J. Amer. Math. Soc. **3** (1990), no. 1, 159–256, With an appendix by the author and C. Soulé.
- [4] ———, *Equivariant immersions and Quillen metrics*, J. Differential Geom. **41** (1995), no. 1, 53–157.
- [5] ———, *Holomorphic families of immersions and higher analytic torsion forms*, Astérisque (1997), no. 244, viii+275.
- [6] J.-M. Bismut, H. Gillet, and C. Soulé, *Analytic torsion and holomorphic determinant bundles. II. Direct images and Bott-Chern forms*, Comm. Math. Phys. **115** (1988), no. 1, 79–126.
- [7] J.-M. Bismut and G. Lebeau, *Complex immersions and Quillen metrics*, Inst. Hautes Études Sci. Publ. Math. (1991), no. 74, ii+298 pp. (1992).
- [8] J.-M. Bismut and É. Vasserot, *The asymptotics of the Ray-Singer analytic torsion associated with high powers of a positive line bundle*, Comm. Math. Phys. **125** (1989), no. 2, 355–367.
- [9] M. Bordemann and E. Meinrenken and M. Schlichenmaier, *Toeplitz quantization of Kähler manifolds and  $gl(N)$ ,  $N \rightarrow \infty$  limits*, Comm. Math. Phys. **165** (1994), no. 2, 281–296.
- [10] D. Borthwick and A. Uribe, *Almost complex structures and geometric quantization*, Math. Res. Lett. **3** (1996), no. 6, 845–861. Erratum: **5** (1998), 211–212.
- [11] L. Boutet de Monvel and V. Guillemin, *The spectral theory of Toeplitz operators*, Annals of Mathematics Studies, vol. 99, Princeton University Press, Princeton, NJ, 1981.
- [12] L. Boutet de Monvel and J. Sjöstrand, *Sur la singularité des noyaux de Bergman et de Szegő*, Journées: Équations aux Dérivées Partielles de Rennes (1975), Soc. Math. France, Paris, 1976, pp. 123–164. Astérisque, No. 34–35.
- [13] M. Braverman, *Vanishing theorems on covering manifolds*, Contemp. Math., vol. 231, Amer. Math. Soc., Providence, RI, 1999, pp. 1–23.
- [14] D. Catlin, *The Bergman kernel and a theorem of Tian*, Analysis and geometry in several complex variables (Katata, 1997), Trends Math., Birkhäuser Boston, Boston, MA, 1999, pp. 1–23.
- [15] L. Charles, *Toeplitz operators and hamiltonian torus action*, J. Funct. Anal. **236** (2006), 299–350.
- [16] J. Chazarain and A. Piriou, *Introduction à la théorie des équations aux dérivées partielles linéaires*, Gauthier-Villars, Paris, 1981.
- [17] X. Dai, K. Liu, and X. Ma, *On the asymptotic expansion of Bergman kernel*, J. Differential Geom. **72** (2006), 1–41; announced in C. R. Math. Acad. Sci. Paris **339** (2004), no. 3, 193–198.
- [18] S. K. Donaldson, *Scalar curvature and projective embeddings. I*, J. Differential Geom. **59** (2001), no. 3, 479–522.
- [19] H. Fegan, *Introduction to compact Lie groups*, Series in Pure Mathematics, vol. 13, World Scientific Publishing Co. Inc., River Edge, NJ, 1991.
- [20] V. Guillemin and S. Sternberg, *Geometric quantization and multiplicities of group representations*, Invent. Math. **67** (1982), no. 3, 515–538.
- [21] ———, *Symplectic techniques in physics*, Cambridge University Press, Cambridge, 1984.

- [22] H. B. Lawson and M.-L. Michelsohn, *Spin geometry*, Princeton Mathematical Series, vol. 38, Princeton University Press, Princeton, NJ, 1989.
- [23] Z. Lu, *On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch*, Amer. J. Math. **122** (2000), no. 2, 235–273.
- [24] X. Ma, *Orbifolds and analytic torsions*, Trans. Amer. Math. Soc. **357** (2005), 2205–2233.
- [25] X. Ma and G. Marinescu, *The  $\text{Spin}^c$  Dirac operator on high tensor powers of a line bundle*, Math. Z. **240** (2002), no. 3, 651–664.
- [26] ———, *Generalized Bergman kernels on symplectic manifolds*, C. R. Math. Acad. Sci. Paris **339** (2004), no. 7, 493–498. The full version: math.DG/0411559.
- [27] ———, *Toeplitz operators on symplectic manifolds*, Preprint.
- [28] ———, *Holomorphic Morse inequalities and Bergman kernels*, Progress in Mathematics, Birkhäuser Boston, Inc., Boston, MA 2007.
- [29] X. Ma and W. Zhang, *Bergman kernels and symplectic reduction*, C. R. Math. Acad. Sci. Paris **341** (2005), 297–302.
- [30] ———, *Toeplitz quantization and symplectic reduction*, Proceeding of 23rd International Conference of Differential Geometric Methods in Theoretical Physics, World Scientific, (2005).
- [31] E. Meinrenken, *On Riemann-Roch formulas for multiplicities*, J. Amer. Math. Soc. **9** (1996), no. 2, 373–389.
- [32] ———, *Symplectic surgery and the  $\text{Spin}^c$ -Dirac operator*, Adv. Math. **134** (1998), no. 2, 240–277.
- [33] R. Paoletti, *Moment maps and equivariant Szegő kernels*, J. Symplectic Geom. **2** (2003), 133–175.
- [34] ———, *The Szegő kernel of a symplectic quotient*, Adv. Math. **197** (2005), 523–553.
- [35] W. Ruan, *Canonical coordinates and Bergmann metrics*, Comm. Anal. Geom. **6** (1998), 589–631.
- [36] B. Shiffman and S. Zelditch, *Asymptotics of almost holomorphic sections of ample line bundles on symplectic manifolds*, J. Reine Angew. Math. **544** (2002), 181–222.
- [37] M. E. Taylor, *Pseudodifferential operators*, Princeton Mathematical Series, vol. 34, Princeton University Press, Princeton, N.J., 1981.
- [38] ———, *Partial differential equations. I, II*, Applied Mathematical Sciences, vol. 115, 116, Springer-Verlag, New York, 1996.
- [39] G. Tian, *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geom. **32** (1990), no. 1, 99–130.
- [40] Y. Tian and W. Zhang, *An analytic proof of the geometric quantization conjecture of Guillemin-Sternberg*, Invent. Math. **132** (1998), no. 2, 229–259.
- [41] M. Vergne, *Multiplicities formula for geometric quantization. I, II*, Duke Math. J. **82** (1996), no. 1, 143–179, 181–194.
- [42] S. Zelditch, *Szegő kernels and a theorem of Tian*, Internat. Math. Res. Notices (1998), no. 6, 317–331.
- [43] W. Zhang, *Holomorphic quantization formula in singular reduction*, Commun. Contemp. Math. **1** (1999), no. 3, 281–293.

CENTRE DE MATHÉMATIQUES LAURENT SCHWARTZ, UMR 7640 DU CNRS, ECOLE POLYTECHNIQUE, 91128 PALAISEAU CEDEX, FRANCE (MA@MATH.POLYTECHNIQUE.FR)

CERN INSTITUTE OF MATHEMATICS & LPMC, NANKAI UNIVERSITY, TIANJIN 300071, P.R. CHINA. (WEIPING@NANKAI.EDU.CN)